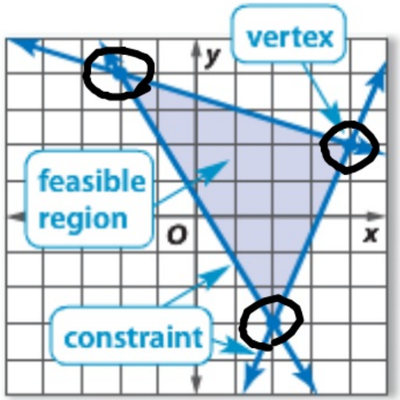
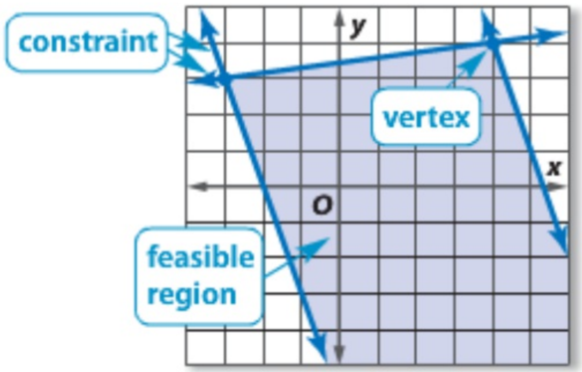
3-3 Optimization with Linear Programming

Linear programming is a method for finding maximum or minimum values of a function over a given system of inequalities with each inequality representing a constraint. After the system is graphed and the vertices of the solution set, called the **feasible region**, are substituted into the function, you can determine the maximum or minimum value.

KeyConcept Feasible Regions



The feasible region is enclosed, or **bounded**, by the constraints. The maximum or minimum value of the related function *always* occurs at a vertex of the feasible region.



The feasible region is open and can go on forever. It is **unbounded**. Unbounded regions have either a maximum or a minimum.

Example 1 Bounded Region

Graph the system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the function for this region.

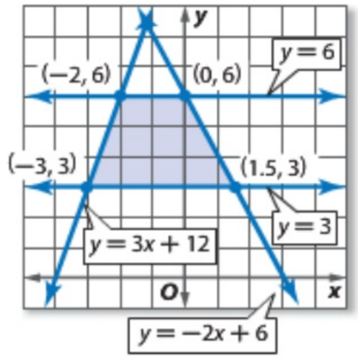
$3 \leq y \leq 6$
$y \leq 3x + 12$
$y \leq -2x + 6$
$f(x, y) = 4x - 2y$

> **Reading Math**
> **Function Notation** The notation $f(x, y)$ is used to represent a function with two variables, x and y. It is read *f of x and y*.

Step 1 Graph the inequalities and locate the vertices.

Step 2 Evaluate the function at each vertex.

(x, y)	$4x - 2y$	$f(x, y)$	
(−3, 3)	4(−3) − 2(3)	−18	
(1.5, 3)	4(1.5) − 2(3)	0	← maximum
(0, 6)	4(0) − 2(6)	−12	
(−2, 6)	4(−2) − 2(6)	−20	← minimum



The maximum value is 0 at (1.5, 3). The minimum value is −20 at (−2, 6).

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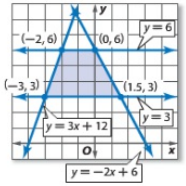

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The maximum value is 0 at (1.5, 3). The minimum value is −20 at (−2, 6).

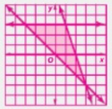
Reading Math

Function Notation The notation $f(x, y)$ is used to represent a function with two variables, x and y. It is read *f of x and y*.

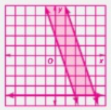
Additional Answers

1. 

(4, 5), (4, −4), (−5, 5); max = 28, min = −35

2. 

(1, 3), (3, −3), (−3, 3); max = 20, min = −12

3. 

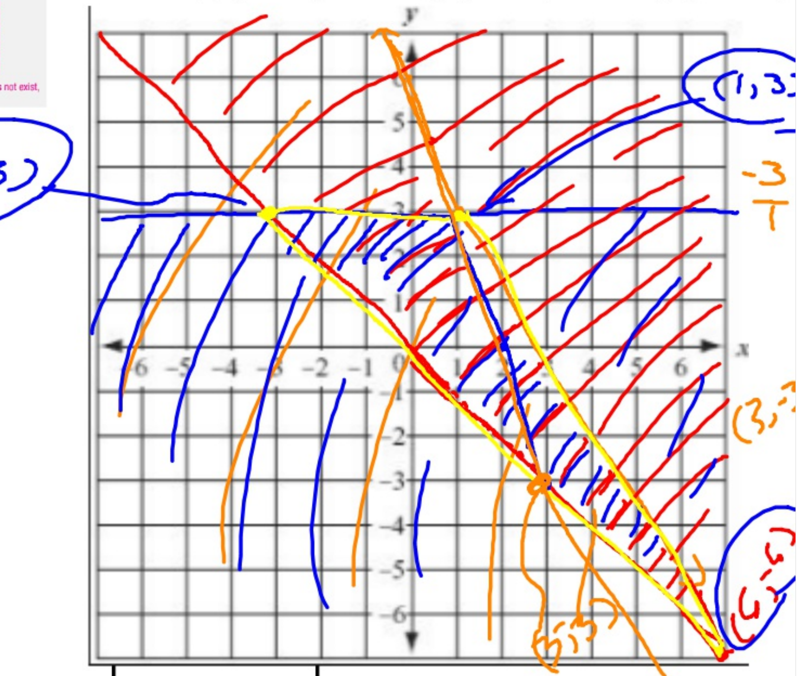
(2, −4), (4, −4); max does not exist, min = −52

Examples 1–2 **Graph each system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the given function for this region.** **1–6. See margin.**

1. $y \le 5$
$x \le 4$
$y \ge -x$
$f(x, y) = 5x - 2y$

2. $y \le -3x + 6$
$-y \le x$ y ≥ −x
$y \le 3$
$f(x, y) = 8x + 4y$

3. $y \ge -3x + 2$
$9x + 3y \le 24$
$y \ge -4$
$f(x, y) = 2x + 14y$



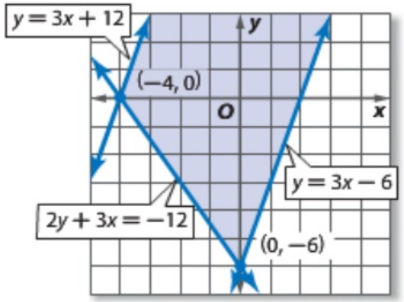
Example 2 Unbounded Region

Graph the system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the function for this region.

$2y + 3x \geq -12$
$y \leq 3x + 12$
$y \geq 3x - 6$
$f(x, y) = 9x - 6y$

Evaluate the function at each vertex.

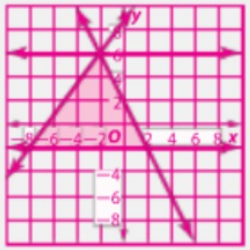
(x, y)	$9x - 6y$	$f(x, y)$
$(-4, 0)$	$9(-4) - 6(0)$	-36
$(0, -6)$	$9(0) - 6(-6)$	36



The maximum value is 36 at (0, −6). There is no minimum value. Notice that another point in the feasible region, (0, 8), yields a value of −48, which is less than −36.

WatchOut!

CCSS Precision Do not assume that there is no maximum if the feasible region is unbounded above the vertices. Test points are needed to determine if there is a minimum or maximum.

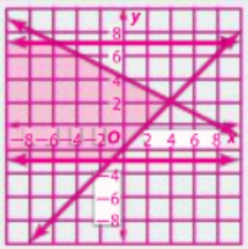
4. 

(2, −2), (−8, −2), (−2, 6); max = 36, min = −30

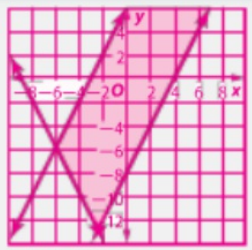
4. $-2 \leq y \leq 6$
$3y \leq 4x + 26$
$y \leq -2x + 2$
$f(x, y) = -3x - 6y$

5. $-3 \leq y \leq 7$
$4y \geq 4x - 8$
$6y + 3x \leq 24$
$f(x, y) = -12x + 9y$

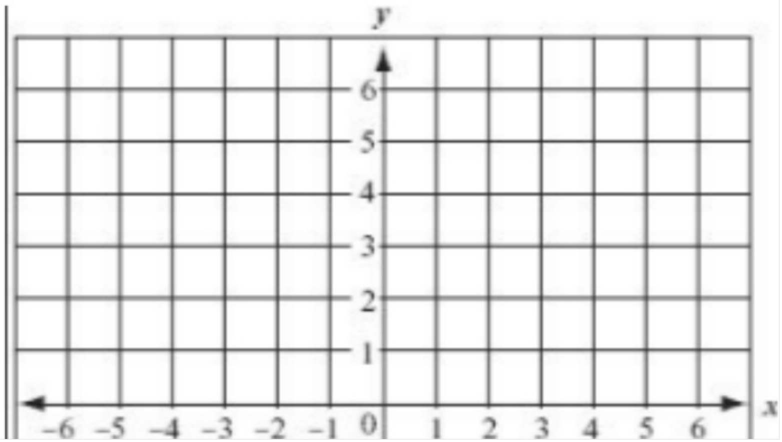
6. $y \leq 2x + 6$
$y \geq 2x - 8$
$y \geq -2x - 18$
$f(x, y) = 5x - 4y$

5. 

(4, 2), (−1, −3), (−6, 7); max does not exist; min = −30

6. 

(−6, −6), (−2.5, −13), no min, max = 39.5



Why?

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An electronics company produces digital audio players and phones. A sign on the company bulletin board is shown.

If at least 2000 items must be produced per shift, how many of each type should be made to minimize costs?

The company is experiencing limitations, or constraints, on production caused by customer demand, shipping, and the productivity of their factory. A system of inequalities can be used to represent these constraints.

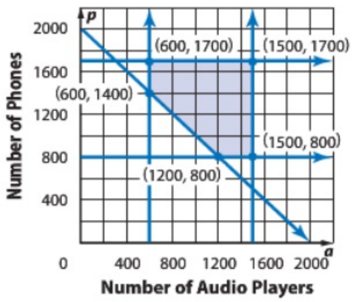
Keeping Costs Down: We Can Do It!

Our Goal: Production per Shift			
Unit	Minimum	Maximum	Cost per Unit
audio	600	1500	$55
phone	800	1700	$95

Real-World Example 3 Optimization with Linear Programming

BUSINESS **Refer to the application at the beginning of the lesson. Determine how many of each type of device should be made per shift.**

Step 1 Let a = number of audio players produced. Let p = number of phones produced.



Step 2

$600 \leq a \leq 1500$

$800 \leq p \leq 1700$

$a + p \geq 2000$

Steps 3 and 4 The system is graphed at the right. Note the vertices of the feasible region.

Step 5 The function to be minimized is $f(a, p) = 55a + 95p$.

Step 6

(a, p)	$55a + 95p$	$f(a, p)$	
(600, 1700)	55(600) + 95(1700)	194,500	
(600, 1400)	55(600) + 95(1400)	166,000	
(1500, 1700)	55(1500) + 95(1700)	244,000	← maximum
(1500, 800)	55(1500) + 95(800)	158,500	
(1200, 800)	55(1200) + 95(800)	142,000	← minimum

Step 7 Produce 1200 audio players and 800 phones to minimize costs.

Example 3

7. **CCSS PRECISION** The total number of workers' hours per day available for production in a skateboard factory is 85 hours. There are 40 hours available for finishing decks and quality control each day. The table shows the number of hours needed in each department for two different types of skateboards.

Skateboard Manufacturing Time		
Board Type	Production Time	Deck Finishing/Quality control
Pro Boards	1.5 hours	2 hours
Specialty Boards	1 hour	0.5 hour

Example 3

7. CCSS **PRECISION** The total number of workers' hours per day available for production in a skateboard factory is 85 hours. There are 40 hours available for finishing decks and quality control each day. The table shows the number of hours needed in each department for two different types of skateboards.

$1.5g + 1c \le 85$

$2g + .5c \le 40$

Skateboard Manufacturing Time		
Board Type	Production Time	Deck Finishing/Quality Control
Pro Boards	1.5 hours	2 hours
Specialty Boards	1 hour	0.5 hour

7a. $g \ge 0$, $c \ge 0$, $1.5g + c \le 85$, $2g + 0.5c \le 40$

a. Write a system of inequalities to represent the situation.

b. Draw the graph showing the feasible region. **See Chapter 3 Answer Appendix.**

c. List the coordinates of the vertices of the feasible region. **(0, 0), (0, 20), (80, 0)**

d. If the profit on a pro board is \$50 and the profit on a specialty board is \$65, write a function for the total profit on the skateboards. $f(c, g) = 65c + 50g$

e. Determine the number of each type of skateboard that needs to be made to have a maximum profit. What is the maximum profit? **80 specialty boards, 0 pro boards; \$5200**

$50g + 65c = $ (profit)

Practice and Problem Solving

Extra Practice is on page R3

Examples 1–2 **Graph each system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the given function for this region.** 8–13. See Chapter 3 Answer Appendix.

8. $1 \leq y \leq 4$
$4y - 6x \geq -32$
$2y \geq -x + 4$
$f(x, y) = -6x + 3y$

9. $2 \geq x \geq -3$
$y \geq -2x - 6$
$4y \leq 2x + 32$
$f(x, y) = -4x - 9y$

10. $-2 \leq x \leq 4$
$5 \leq y \leq 8$
$2x + 3y \leq 26$
$f(x, y) = 8x - 10y$

11. $-8 \leq y \leq -2$
$y \leq x$
$y \leq -3x + 10$
$f(x, y) = 5x + 14y$

12. $x + 4y \geq 2$
$2x + 4y \leq 24$
$2 \leq x \leq 6$
$f(x, y) = 6x + 7y$

13. $3 \leq y \leq 7$
$2y + x \leq 8$
$y - 2x \leq 23$
$f(x, y) = -3x + 5y$

nples 1–2 **Graph each system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the given function for this region.** 14–22. See margin.

14. $-9 \le x \le -3$
$-9 \le y \le -5$
$3y + 12x \le -75$
$f(x, y) = 20x + 8y$

15. $x \ge -8$
$3x + 6y \le 36$
$2y + 12 \ge 3x$
$f(x, y) = 10x - 6y$

16. $y \ge |x - 2|$
$y \le 8$
$8y + 5x \le 49$
$f(x, y) = -5x - 15y$

17. $x \ge -6$
$y + x \le -1$
$2x + 3y \ge -9$
$f(x, y) = -10x - 12y$

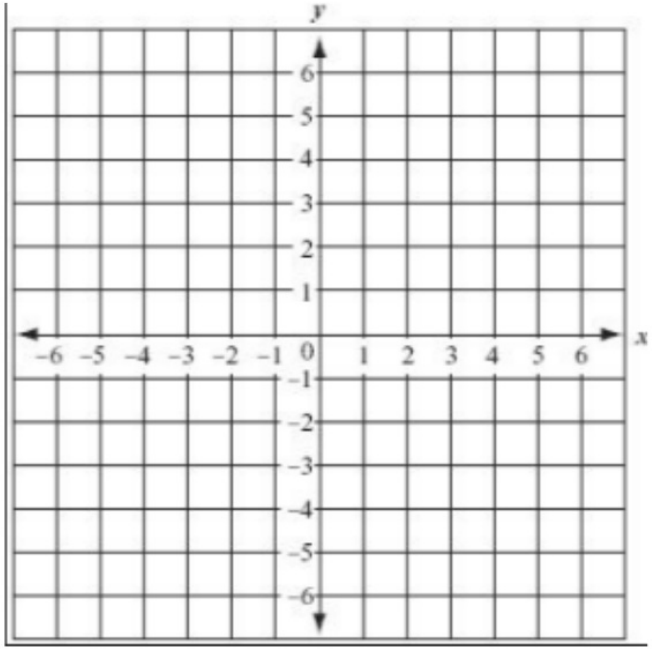
18. $-5 \ge y \ge -17$
$y \le 3x + 19$
$y \le -4x + 15$
$f(x, y) = 8x - 3y$

19. $-8 \le x \le 16$
$y \ge 2x - 10$
$2y + x \le 80$
$f(x, y) = 12x + 15y$

20. $y \le x + 4$
$y \ge x - 4$
$y \le -x + 10$
$y \ge -x - 10$
$f(x, y) = -10x + 9y$

21. $-4 \le x \le 8$
$-8 \le y \le 6$
$y \ge x - 6$
$4y + 7x \le 31$
$f(x, y) = 12x + 8y$

22. $y \ge |x + 1| - 2$
$0 \le y \le 6$
$-6 \le x \le 2$
$x + 3y \le 14$
$f(x, y) = 5x + 4y$



Additional Answers

14–22. See Ch. 3 Answer Appendix for graphs.

14. (−9, −9), (−4, −9), (−5, −5), (−9, −5); max = −140, min = −252

15. (6, 3), (−8, 10), (−8, −18); max = 42, min = −140

16. (2, 0), (5, 3), (−3, 8), (−6, 8); max = −10, min = −105

17. (−6, 1), (6, −7), (−6, 5); max = 48, min = 0

18. (5, −5), (8, −17), (−12, −17), (−8, −5); max = 115, min = −49

19. (−8, 44), (16, 32), (−8, −26), (16, 22); max = 672, min = −486

20. (3, 7), (7, 3), (−3, −7), (−7, −3); max = 43, min = −43

21. (5, −1), (1, 6), (−2, −8), (−4, −8), (−4, 6); max = 60, min = −112

22. (−4, 6), (2, 4), (2, 1), (1, 0), (−3, 0), (−6, 3), (−6, 6); max = 26, min = −18

Example 3

23. COOKING Jenny's Bakery makes two types of birthday cakes: yellow cake, which sells for $25, and strawberry cake, which sells for $35. Both cakes are the same size, but the decorating and assembly time required for the yellow cake is 2 hours, while the time is 3 hours for the strawberry cake. There are 450 hours of labor available for production. How many of each type of cake should be made to maximize revenue? **225 yellow cakes, 0 strawberry cakes**

24. BUSINESS The manager of a travel agency is printing brochures and fliers to advertise special discounts on vacation spots during the summer months. Each brochure costs $0.08 to print, and each flier costs $0.04 to print. A brochure requires 3 pages, and a flier requires 2 pages. The manager does not want to use more than 600 pages, and she needs at least 50 brochures and 150 fliers. How many of each should she print to minimize the cost? **50 brochures, 150 fliers**