3-6 Multiplying Matrices

1 Multiply Matrices You can multiply two matrices A and B if and only if the number of columns in A is equal to the number of rows in B. When you multiply two matrices $A_{m \times r}$ and $B_{r \times t}$, the resulting matrix AB is an $m \times t$ matrix.

A B $=$ AB

$m \times r$ $r \times t$ $m \times t$

same

dimensions of AB

rows x columns

Example 1 Dimensions of Matrix Products

Determine whether each matrix product is defined. If so, state the dimensions of the product.

a. $A_{3 \times 4}$ **and** $B_{4 \times 2}$

$A \cdot B = AB$

3×4 4×2 3×2

The inner dimensions are equal, so the product is defined. Its dimensions are 3×2.

b. $A_{5 \times 3}$ **and** $B_{5 \times 4}$

$A \cdot B$

5×3 5×4

Example 1 **Determine whether each matrix product is defined. If so, state the dimensions of the product.**

1. $A_{2 \times 4} \cdot B_{4 \times 3}$ **2 × 3**

2. $C_{5 \times 4} \cdot D_{5 \times 4}$ **undefined**

3. $E_{8 \times 6} \cdot F_{6 \times 10}$ **8 × 10**

KeyConcept Multiplying Matrices

Words The element in the *m*th row and *r*th column of matrix *AB* is the sum of the pro
the corresponding elements in row *m* of matrix *A* and column *r* of matrix *B*.

Symbols

$$\begin{matrix} A & \cdot & B & = & AB \end{matrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

$$XY = \begin{bmatrix} 6 & -3 \\ -10 & -2 \end{bmatrix} \cdot \begin{bmatrix} -5 & -4 \\ 3 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -3 \\ -10 & -2 \end{bmatrix} \cdot \begin{bmatrix} -5 & -4 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 6(-5)+(-3)(3) & 6(-4)+(-3)(3) \\ -10(-5)+(-2)(3) & -10(-4)+(-2)(3) \end{bmatrix}$$

Simplify the product matrix.

$$\begin{bmatrix} 6(-5)+(-3)(3) & 6(-4)+(-3)(3) \\ -10(-5)+(-2)(3) & -10(-4)+(-2)(3) \end{bmatrix} = \begin{bmatrix} -39 & -33 \\ 44 & 34 \end{bmatrix}$$

KeyConcept Multiplying Matrices

Words The element in the *m*th row and *r*th column of matrix *AB* is the sum of the products of the corresponding elements in row *m* of matrix *A* and column *r* of matrix *B*.

Symbols

$$\underset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \cdot \underset{B}{\begin{bmatrix} e & f \\ g & h \end{bmatrix}} = \underset{AB}{\begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}}$$

Examples 2–3 **Find each product, if possible.**

4. $\begin{bmatrix} 2 & 1 \\ 7 & -5 \end{bmatrix} \cdot \begin{bmatrix} -6 & 3 \\ -2 & -4 \end{bmatrix}$ $\begin{bmatrix} -14 & 2 \\ -32 & 41 \end{bmatrix}$

5. $\begin{bmatrix} 10 & -2 \\ -7 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 \\ 5 & -2 \end{bmatrix}$ $\begin{bmatrix} 0 & 44 \\ 8 & -34 \end{bmatrix}$

6. $\begin{bmatrix} 9 & -2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 4 \\ 6 & -7 \end{bmatrix}$ $\begin{bmatrix} -30 & 50 \end{bmatrix}$

7. $\begin{bmatrix} -9 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} -1 & -10 & 1 \end{bmatrix}$ $\begin{bmatrix} 9 & 90 & -9 \\ -6 & -60 & 6 \end{bmatrix}$

8. $\begin{bmatrix} -8 & 7 & 4 \\ -5 & -3 & 8 \end{bmatrix} \cdot \begin{bmatrix} 10 & 6 \\ 8 & 4 \end{bmatrix}$ undefined

9. $\begin{bmatrix} 2 & 8 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -7 \end{bmatrix}$ $\begin{bmatrix} -44 \\ 25 \end{bmatrix}$

10. $\begin{bmatrix} -4 & 3 & 2 \\ -1 & -5 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 6 \\ 8 & 4 & -1 \\ 5 & 3 & -2 \end{bmatrix}$ $\begin{bmatrix} 26 & 14 & -31 \\ -22 & -9 & -9 \end{bmatrix}$

11. $\begin{bmatrix} 2 & 5 & 3 & -1 \\ -3 & 1 & 8 & -3 \end{bmatrix} \cdot \begin{bmatrix} 6 & -3 \\ -7 & 1 \\ 2 & 0 \\ -1 & 0 \end{bmatrix}$ $\begin{bmatrix} -16 & -1 \\ -6 & 10 \end{bmatrix}$

$2\times1 \;\; 1\times3 = 2\times3$

$2\times3 \;\; 2\times2$

$2\times3 \;\; 3\times3$

$(-4)(2) + (3)(8) + (2)(5)$

$-8 + 24 + 10$

6. $[9 \quad -2] \cdot \begin{bmatrix} -2 & 4 \\ 6 & -7 \end{bmatrix}$ **[−30   50]**

1×2   3×2   1×2

9(−2) + (−2)(6)   36+14

−18 + −12

7 $\begin{bmatrix} -9 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} -1 & -10 & 1 \end{bmatrix}$ $\begin{bmatrix} 9 & 90 & -9 \\ -6 & -60 & 6 \end{bmatrix}$

$(-9)(-1)$ $(-9)(-10)$ $(-9)(1)$

$(6)(-1)$ $(6)(-10)$ $(6)(1)$

$12 + -56$

$18 + 7$

2×2 2×1 2×1

9. $\begin{bmatrix} 2 & 8 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -7 \end{bmatrix}$ $\begin{bmatrix} -44 \\ 25 \end{bmatrix}$

$(2)(6) + 8(-7)$

$(3)(6) + (-1)(-7)$

SWIM MEET **At a particular swim meet, 7 points were awarded for each first-place finish, 4 points for second, and 2 points for third. Find the total number of points for each school. Which school won the meet?**

School	First Place	Second Place	Third Place
Central	4	7	3
Franklin	8	9	1
Hayes	10	5	3
Lincoln	3	3	6

Understand The final scores can be found by multiplying the swim results for each school by the points awarded for each first-, second-, and third-place finish.

Plan Write the results of the races and the points awarded in matrix form. Set up the matrices so that the number of rows in the points matrix equals the number of columns in the results matrix.

Results **Points**

$$R = \begin{bmatrix} 4 & 7 & 3 \\ 8 & 9 & 1 \\ 10 & 5 & 3 \\ 3 & 3 & 6 \end{bmatrix} \qquad P = \begin{bmatrix} 7 \\ 4 \\ 2 \end{bmatrix}$$

Solve Multiply the matrices.

$$RP = \begin{bmatrix} 4 & 7 & 3 \\ 8 & 9 & 1 \\ 10 & 5 & 3 \\ 3 & 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \\ 2 \end{bmatrix}$$ Write an equation.

$$= \begin{bmatrix} 4(7) + 7(4) + 3(2) \\ 8(7) + 9(4) + 1(2) \\ 10(7) + 5(4) + 3(2) \\ 3(7) + 3(4) + 6(2) \end{bmatrix}$$ Multiply columns by rows.

$$= \begin{bmatrix} 62 \\ 94 \\ 96 \\ 45 \end{bmatrix}$$ Simplify.

12. **SENSE-MAKING** The table shows the number of people registered for aerobics for the first quarter. **a. See margin.**

Quinn's Gym charges the following registration fees: class-by-class, \$165; 11-class pass, \$110; unlimited pass, \$239.

Quinn's Gym		
Payment	**Aerobics**	**Step Aerobics**
class-by-class	35	28
11-class pass	32	17
unlimited pass	18	12

a. Write a matrix for the registration fees and a matrix for the number of students.

b. Find the total amount of money the gym received from aerobics and step aerobic registrations. **\$22,955**

Example 4 Test of the Commutative Property

Find each product if $G = \begin{bmatrix} 1 & 3 & -5 \\ 4 & -2 & 0 \end{bmatrix}$ **and** $H = \begin{bmatrix} 2 & 3 \\ -2 & -8 \\ 1 & 7 \end{bmatrix}$.

a. GH

$$GH = \begin{bmatrix} 1 & 3 & -5 \\ 4 & -2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ -2 & -8 \\ 1 & 7 \end{bmatrix}$$ Substitution

$$= \begin{bmatrix} 2 - 6 - 5 & 3 - 24 - 35 \\ 8 + 4 + 0 & 12 + 16 + 0 \end{bmatrix} \text{ or } \begin{bmatrix} -9 & -56 \\ 12 & 28 \end{bmatrix}$$

b. HG

$$HG = \begin{bmatrix} 2 & 3 \\ -2 & -8 \\ 1 & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & -5 \\ 4 & -2 & 0 \end{bmatrix}$$ Substitution

$$= \begin{bmatrix} 2 + 12 & 6 - 6 & -10 + 0 \\ -2 - 32 & -6 + 16 & 10 + 0 \\ 1 + 28 & 3 - 14 & -5 + 0 \end{bmatrix} \text{ or } \begin{bmatrix} 14 & 0 & -10 \\ -34 & 10 & 10 \\ 29 & -11 & -5 \end{bmatrix}$$

Notice that $GH \neq HG$.

Example 5 Test of the Distributive Property

Find each product if $J = \begin{bmatrix} 2 & 4 \\ -5 & -2 \end{bmatrix}$, $K = \begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix}$, **and** $L = \begin{bmatrix} -4 & -1 \\ 3 & 0 \end{bmatrix}$.

a. $J(K + L)$

$$J(K + L) = \begin{bmatrix} 2 & 4 \\ -5 & -2 \end{bmatrix} \cdot \left(\begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} -4 & -1 \\ 3 & 0 \end{bmatrix} \right)$$ Substitution

$$= \begin{bmatrix} 2 & 4 \\ -5 & -2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$$ Add.

$$= \begin{bmatrix} -2 + 8 & 2 + 12 \\ 5 - 4 & -5 - 6 \end{bmatrix} \text{ or } \begin{bmatrix} 6 & 14 \\ 1 & -11 \end{bmatrix}$$ Multiply.

b. $JK + JL$

$$JK + JL = \begin{bmatrix} 2 & 4 \\ -5 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ -5 & -2 \end{bmatrix} \cdot \begin{bmatrix} -4 & -1 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2(3) + 4(-1) & 2(2) + 4(3) \\ -5(3) + (-2)(-1) & -5(2) + (-2)(3) \end{bmatrix} + \begin{bmatrix} 2(-4) + 4(3) & 2(-1) + 4(0) \\ -5(-4) + (-2)(3) & -5(-1) + (-2)(0) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 16 \\ -13 & -16 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ 14 & 5 \end{bmatrix} \text{ or } \begin{bmatrix} 6 & 14 \\ 1 & -11 \end{bmatrix}$$

Notice that $J(K + L) = JK + JL$.

Examples 4–5 Use $X = \begin{bmatrix} -10 & -3 \\ 2 & -8 \end{bmatrix}$, $Y = \begin{bmatrix} -5 & 6 \\ -1 & 9 \end{bmatrix}$, and $Z = \begin{bmatrix} -5 & -1 \\ -8 & -4 \end{bmatrix}$ to determine whether the following equations are true for the given matrices. **13, 14. See margin.**

13. $XY = YX$

14. $X(YZ) = (XY)Z$

13. No; $\begin{bmatrix} 53 & -87 \\ -2 & -60 \end{bmatrix} \neq \begin{bmatrix} 62 & -33 \\ 28 & -69 \end{bmatrix}$.

14. Yes; $X(YZ) = \begin{bmatrix} 431 & 295 \\ 490 & 242 \end{bmatrix}$ and $(XY)Z = \begin{bmatrix} 431 & 295 \\ 490 & 242 \end{bmatrix}$.

$$\overset{X}{\begin{bmatrix} -10 & -3 \\ 2 & -8 \end{bmatrix}} \overset{Y}{\begin{bmatrix} -5 & 6 \\ -1 & 9 \end{bmatrix}}$$

$$\begin{bmatrix} (-10)(-5)+(-3)(-1) & (-10)(6)+(-3)(9) \\ (-5)(2)+(-8)(-1) & (2)(6)+(-8)(9) \end{bmatrix} =$$

Practice and Problem Solving

Extra Practice is on page R3.

Example 1 Determine whether each matrix product is defined. If so, state the dimensions of the product.

15. $P_{2 \times 3} \cdot Q_{3 \times 4}$ **2 × 4**

16. $A_{5 \times 5} \cdot B_{5 \times 5}$ **5 × 5**

17. $M_{3 \times 1} \cdot N_{2 \times 3}$ **undefined**

18. $X_{2 \times 6} \cdot Y_{6 \times 3}$ **2 × 3**

19. $J_{2 \times 1} \cdot K_{2 \times 1}$ **undefined**

20. $S_{5 \times 2} \cdot T_{2 \times 4}$ **5 × 4**

Examples 2–3 Find each product, if possible.

21. $\begin{bmatrix} 1 & 6 \end{bmatrix} \cdot \begin{bmatrix} -10 \\ 6 \end{bmatrix}$ **$[26]$**

22. $\begin{bmatrix} 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 2 & -7 \end{bmatrix}$ $\begin{bmatrix} 12 & -42 \\ -6 & 21 \end{bmatrix}$

23. $\begin{bmatrix} -3 & -7 \\ -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 4 \\ 9 & -3 \end{bmatrix}$ $\begin{bmatrix} -75 & 9 \\ -17 & -5 \end{bmatrix}$

24. $\begin{bmatrix} -1 & 0 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 6 & -3 \\ 7 & -2 \end{bmatrix}$ $\begin{bmatrix} -6 & 3 \\ 44 & -19 \end{bmatrix}$

25. $\begin{bmatrix} -1 & 0 & 6 \\ -4 & -10 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & -7 \\ -2 & -9 \end{bmatrix}$ **undefined**

26. $\begin{bmatrix} -6 & 4 & -9 \\ 2 & 8 & 7 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 2 \\ 4 \end{bmatrix}$ $\begin{bmatrix} -70 \\ 58 \end{bmatrix}$

27. $\begin{bmatrix} 2 & 9 & -3 \\ 4 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 & 2 \\ -6 & 7 \\ -2 & 1 \end{bmatrix}$ $\begin{bmatrix} -40 & 64 \\ 22 & 1 \end{bmatrix}$

28. $\begin{bmatrix} -4 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} -3 & -1 \end{bmatrix}$ $\begin{bmatrix} 12 & 4 \\ -24 & -8 \end{bmatrix}$

29. **TRAVEL** The Wolf family owns three bed and breakfasts in a vacation spot. A room with a single bed is \$220 per night, a room with two beds is \$250 per night, and a suite is \$360.

a. Write a matrix for the number of each type of room at each bed and breakfast. Then write a room-cost matrix. **See margin.**

b. Write a matrix for total daily income, assuming that all the rooms are rented.

29b. $\begin{bmatrix} \$1880 \\ \$1550 \\ \$1630 \end{bmatrix}$

Available Rooms at a Wolf Bed and Breakfast			
B & B	Single	Double	Suite
1	3	2	2
2	2	3	1
3	4	3	0

c. What is the total daily income from all three bed and breakfasts, assuming that all the rooms are rented? **\$5060**

Examples 4–5 **Use $P = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$, $Q = \begin{bmatrix} 6 & 4 \\ -2 & -5 \end{bmatrix}$, $R = \begin{bmatrix} 4 & 6 \\ -6 & 4 \end{bmatrix}$, and $k = 2$ to determine whether the following equations are true for the given matrices.** **30–33. See margin.**

30. $k(PQ) = P(kQ)$

31. $PQR = RQP$

32. $PR + QR = (P + Q)R$

33. $R(P + Q) = PR + QR$

Additional Answers

29a. $I = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 1 \\ 4 & 3 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 220 \\ 250 \\ 360 \end{bmatrix}$

30. Yes; $k(PQ) = \begin{bmatrix} 52 & 42 \\ 4 & -12 \end{bmatrix}$ and $P(kQ) = \begin{bmatrix} 52 & 42 \\ 4 & -12 \end{bmatrix}$.

31. No; $PQR = \begin{bmatrix} -22 & 240 \\ 44 & -12 \end{bmatrix}$ and $RQP = \begin{bmatrix} 34 & -40 \\ -220 & -44 \end{bmatrix}$.

32. Yes; $PR + QR = \begin{bmatrix} 22 & 72 \\ 14 & -18 \end{bmatrix}$ and $(P + Q)R = \begin{bmatrix} 22 & 72 \\ 14 & -18 \end{bmatrix}$.

33. No; $R(P + Q) = \begin{bmatrix} 34 & -6 \\ -64 & -30 \end{bmatrix}$ and $PR + QR = \begin{bmatrix} 22 & 72 \\ 14 & -18 \end{bmatrix}$.