3-7 Solving Systems of Equations using Cramer's Rule

1 Determinants Every square matrix has a **determinant**. The determinant of a 2×2 matrix is called a **second-order determinant**.

KeyConcept Second-Order Determinant

Words The value of a second-order determinant is the difference of the products of the two diagonals.

Symbols $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Example $\begin{vmatrix} 4 & 5 \\ -3 & 6 \end{vmatrix} = 4(6) - (5)(-3) = 39$

Example 1 Second-Order Determinant

Evaluate each determinant.

a. $\begin{vmatrix} 5 & -4 \\ 8 & 9 \end{vmatrix}$

$\begin{vmatrix} 5 & -4 \\ 8 & 9 \end{vmatrix} = 5(9) - (-4)(8)$ Definition of determinant

$= 45 + 32$ Simplify.

$= 77$

b. $\begin{vmatrix} 0 & 6 \\ 4 & -11 \end{vmatrix}$

$\begin{vmatrix} 0 & 6 \\ 4 & -11 \end{vmatrix} = 0(-11) - 6(4)$ Definition of determinant

$= 0 - 24$ Simplify.

$= -24$

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Example $\begin{vmatrix} 4 & 5 \\ -3 & 6 \end{vmatrix} = 4(6) - (5)(-3) = 39$

$16x - 10y = \sim$
$-8x + 5y = \sim$

Example 1 **Evaluate each determinant.**

1. $\begin{vmatrix} 8 & 6 \\ 5 & 7 \end{vmatrix}$ **26**

2. $\begin{vmatrix} -6 & -6 \\ 8 & 10 \end{vmatrix}$ **−12**

3. $\begin{vmatrix} -4 & 12 \\ 9 & 5 \end{vmatrix}$ **−128**

4. $\begin{vmatrix} 16 & -10 \\ -8 & 5 \end{vmatrix}$ **0**

① $(8)(7) - (5)(6)$
$56 - 30 = 26$

$(16)(5) - (-8)(-10)$
$80 - 80$

Step 1 Rewrite the first two columns to the right of the determinant.

$$\left|\begin{array}{rrr} 4 & -8 & 3 \\ -3 & 2 & 6 \\ -4 & 5 & 9 \end{array}\right|\begin{array}{rr} 4 & -8 \\ -3 & 2 \\ -4 & 5 \end{array}$$

Step 2 Find the products of the elements of the diagonals.

$$\left|\begin{array}{rrr} 4 & -8 & 3 \\ -3 & 2 & 6 \\ -4 & 5 & 9 \end{array}\right|\begin{array}{rr} 4 & -8 \\ -3 & 2 \\ -4 & 5 \end{array} \qquad \left|\begin{array}{rrr} 4 & -8 & 3 \\ -3 & 2 & 6 \\ -4 & 5 & 9 \end{array}\right|\begin{array}{rr} 4 & -8 \\ -3 & 2 \\ -4 & 5 \end{array}$$

$4(2)(9) = 72$

$-8(6)(-4) = 192$

$3(-3)(5) = -45$

$-4(2)(3) = -24$

$5(6)(4) = 120$

$9(-3)(-8) = 216$

Step 3 Find the sum of each group.

$72 + 192 + (-45) = 219$

$-24 + 120 + 216 = 312$

Step 4 Subtract the sum of the second group from the sum of the first group.

$219 - 312 = -93$

The value of the determinant is -93.

Example 2 Evaluate each determinant using diagonals.

5 $\begin{vmatrix} 3 & -2 & 2 \\ -4 & 2 & -5 \\ -3 & 1 & 4 \end{vmatrix}$ **−19**

(5)

$\begin{vmatrix} 3 & -2 & 2 \\ -4 & 2 & -5 \\ -3 & 1 & 4 \end{vmatrix} \begin{matrix} 3 & -2 \\ -4 & 2 \\ -3 & 1 \end{matrix}$

$3 \times 2 \times 4 = 24$

$-2 \times -5 \times -3 = -30$

$= -8$

-14

$2 \times 2 \times -3 = -12$

$= -15$

$= 32$

5

Example 2 **Evaluate each determinant using diagonals.**

5. $\begin{vmatrix} 3 & -2 & 2 \\ -4 & 2 & -5 \\ -3 & 1 & 4 \end{vmatrix}$ **−19**

6. $\begin{vmatrix} 2 & -3 & 5 \\ -4 & 6 & -2 \\ 4 & -1 & -6 \end{vmatrix}$ **−80**

7. $\begin{vmatrix} 8 & 4 & 0 \\ -2 & -6 & -1 \\ 5 & -3 & 6 \end{vmatrix}$ **−284**

8. $\begin{vmatrix} -5 & -3 & 4 \\ -2 & -4 & -3 \\ 8 & -2 & 4 \end{vmatrix}$ **302**

9. $\begin{vmatrix} 8 & 3 & 4 \\ 2 & 4 & 2 \\ 1 & 6 & 5 \end{vmatrix}$ **72**

10. $\begin{vmatrix} -4 & 3 & 0 \\ 1 & 5 & -2 \\ -1 & -8 & -3 \end{vmatrix}$ **139**

11. $\begin{vmatrix} 2 & -6 & -3 \\ 7 & 9 & -4 \\ -6 & 4 & 9 \end{vmatrix}$ **182**

12. $\begin{vmatrix} -5 & -6 & 7 \\ 4 & 0 & 5 \\ -3 & 8 & 2 \end{vmatrix}$ **562**

Examples 1–2 **Evaluate each determinant.**

26. $\begin{vmatrix} -7 & 12 \\ 5 & 6 \end{vmatrix}$ **−102**

27. $\begin{vmatrix} -8 & -9 \\ 11 & 12 \end{vmatrix}$ **3**

28. $\begin{vmatrix} -5 & 8 \\ -6 & -7 \end{vmatrix}$ **83**

29. $\begin{vmatrix} 3 & 5 & -2 \\ -1 & -4 & 6 \\ -6 & -2 & 5 \end{vmatrix}$ **−135**

30. $\begin{vmatrix} 2 & 0 & -6 \\ -3 & -4 & -5 \\ -2 & 5 & 8 \end{vmatrix}$ **124**

31. $\begin{vmatrix} -5 & -1 & -2 \\ 1 & 8 & 4 \\ 0 & -6 & 9 \end{vmatrix}$ **−459**

32. $\begin{vmatrix} 6 & -3 & -5 \\ 0 & -7 & 0 \\ 3 & -6 & -4 \end{vmatrix}$ **63**

33. $\begin{vmatrix} -8 & -3 & -9 \\ 0 & 0 & 0 \\ 8 & -2 & -4 \end{vmatrix}$ **0**

34. $\begin{vmatrix} 1 & 6 & 7 \\ -2 & -5 & -8 \\ 4 & 4 & 9 \end{vmatrix}$ **−13**

35. $\begin{vmatrix} 1 & -8 & -9 \\ 6 & 5 & -6 \\ -2 & -8 & 10 \end{vmatrix}$ **728**

36. $\begin{vmatrix} 5 & -5 & -5 \\ -8 & -3 & -2 \\ -2 & 4 & 6 \end{vmatrix}$ **−120**

37. $\begin{vmatrix} -4 & 1 & -2 \\ 10 & 12 & 9 \\ -6 & 0 & 13 \end{vmatrix}$ **−952**

KeyConcept Cramer's Rule

Let C be the coefficient matrix of the system $\begin{matrix} ax + by = m \\ fx + gy = n \end{matrix} \rightarrow \begin{bmatrix} a & b \\ f & g \end{bmatrix}$.

The solution of this system is $x = \dfrac{\begin{vmatrix} m & b \\ n & g \end{vmatrix}}{|C|}$ and $y = \dfrac{\begin{vmatrix} a & m \\ f & n \end{vmatrix}}{|C|}$, if $|C| \neq 0$.

Example 4 Solve a System of Two Equations

Solve the system by using Cramer's Rule.

$5x - 6y = 15$

$3x + 4y = -29$

Example 4 **Use Cramer's Rule to solve each system of equations.**

13. $4x - 5y = 39$ **(6, −3)**
$3x + 8y = -6$

14. $5x + 6y = 20$ **(−2, 5)**
$-3x - 7y = -29$

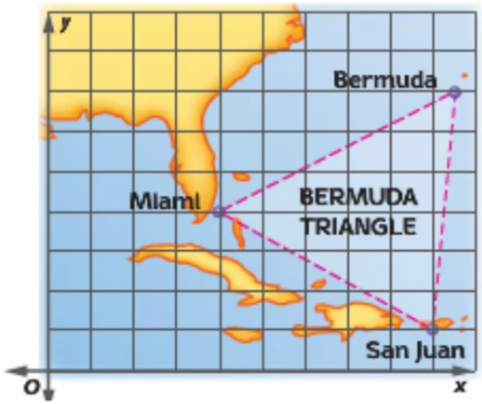
15. $-8a - 5b = -27$ **(4, −1)**
$7a + 6b = 22$

16. $10c - 7d = -59$ **(−8, −3)**
$6c + 5d = -63$

Examples 3–5 **17.** CCSS **PERSEVERANCE** The "Bermuda Triangle" is an area located off the southeastern Atlantic coast of the United States, and is noted for reports of unexplained losses of ships, small boats, and aircraft.

a. Find the area of the triangle on the map. **15.75 units²**

b. Suppose each grid represents 175 miles. What is the area of the Bermuda Triangle? **482,343.75 mi²**



KeyConcept Cramer's Rule for a System of Three Equations

Let C be the coefficient matrix of the system

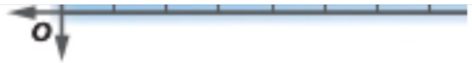
$$\begin{aligned} ax + by + cz &= m \\ fx + gy + hz &= n \\ jx + ky + \ell z &= p \end{aligned} \rightarrow \begin{bmatrix} a & b & c \\ f & g & h \\ j & k & \ell \end{bmatrix}.$$

The solution of this system is $x = \dfrac{\begin{vmatrix} m & b & c \\ n & g & h \\ p & k & \ell \end{vmatrix}}{|C|}$, $y = \dfrac{\begin{vmatrix} a & m & c \\ f & n & h \\ j & p & \ell \end{vmatrix}}{|C|}$, and $z = \dfrac{\begin{vmatrix} a & b & m \\ f & g & n \\ j & k & p \end{vmatrix}}{|C|}$, if $|C| \neq 0$.

Example 5 Solve a System of Three Equations

Solve the system by using Cramer's Rule.

$4x + 5y - 6z = -14$
$3x - 2y + 7z = 47$
$7x - 6y - 8z = 15$



Use Cramer's Rule to solve each system of equations.

18. $4x - 2y + 7z = 26$ **(−3, −5, 4)**
$5x + 3y - 5z = -50$
$-7x - 8y - 3z = 49$

19. $-3x - 5y + 10z = -4$ $\left(\frac{66}{7}, -\frac{116}{7}, -\frac{41}{7}\right)$
$-8x + 2y - 3z = -91$
$6x + 8y - 7z = -35$

20. $6x - 5y + 2z = -49$ **(−3, 7, 2)**
$-5x - 3y - 8z = -22$
$-3x + 8y - 5z = 55$

21. $-9x + 5y + 3z = 50$ **(−4, −2, 8)**
$7x + 8y - 2z = -60$
$-5x + 7y + 5z = 46$

22. $x + 2y = 12$ **(6, 3, −4)**
$3y - 4z = 25$
$x + 6y + z = 20$

23. $9a + 7b = -30$ **(−1, −3, 7)**
$8b + 5c = 11$
$-3a + 10c = 73$

24. $2n + 3p - 4w = 20$ **(5, 2, −1)**
$4n - p + 5w = 13$
$3n + 2p + 4w = 15$

25. $x + y + z = 12$ **(4, 0, 8)**
$6x - 2y - z = 16$
$3x + 4y + 2z = 28$

Examples 4–5 **Use Cramer's Rule to solve each system of equations.**

39. $6x - 5y = 73$ **(8, −5)**
$-7x + 3y = -71$

40. $10a - 3b = -34$ **(−4, −2)**
$3a + 8b = -28$

41. $-4c - 5d = -39$ **(6, 3)**
$5c + 8d = 54$

42. $-6f - 8g = -22$ **(5, −1)**
$-11f + 5g = -60$

43. $9r + 4s = -55$ **(−3, −7)**
$-5r - 3s = 36$

44. $-11u - 7v = 4$ **(−8, 12)**
$9u + 4v = -24$

45. $5x - 4y + 6z = 58$ **(4, −2, 5)**
$-4x + 6y + 3z = -13$
$6x + 3y + 7z = 53$

46. $8x - 4y + 7z = 34$ **(−3, −4, 6)**
$5x + 6y + 3z = -21$
$3x + 7y - 8z = -85$

47. **DOUGHNUTS** Mi-Ling is ordering doughnuts for a class party. The box contains 2 dozen doughnuts, some of which are plain and some of which are jelly-filled. The plain doughnuts each cost \$0.50, and the jelly-filled cost \$0.60. If the total cost is \$12.60, use Cramer's Rule to find how many jelly-filled doughnuts Mi-Ling ordered. **6**