3-7 Solving Systems of Equations using Cramer's Rule

1 Determinants Every square matrix has a **determinant**. The determinant of a 2×2 matrix is called a **second-order determinant**.

KeyConcept Second-Order Determinant

Words The value of a second-order determinant is the difference of the products of the two diagonals.

Symbols $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Example $\begin{vmatrix} 4 & 5 \\ -3 & 6 \end{vmatrix} = 4(6) - (5)(-3) = 39$

Example 1 Second-Order Determinant

Evaluate each determinant.

a. $\begin{vmatrix} 5 & -4 \\ 8 & 9 \end{vmatrix}$

$$\begin{vmatrix} 5 & -4 \\ 8 & 9 \end{vmatrix} = 5(9) - (-4)(8)$$ Definition of determinant

$$= 45 + 32$$ Simplify.

$$= 77$$

b. $\begin{vmatrix} 0 & 6 \\ 4 & -11 \end{vmatrix}$

$$\begin{vmatrix} 0 & 6 \\ 4 & -11 \end{vmatrix} = 0(-11) - 6(4)$$ Definition of determinant

$$= 0 - 24$$ Simplify.

$$= -24$$

KeyConcept Second-Order Determinant

Words: The value of a second-order determinant is the difference of the products of the two diagonals.

Symbols: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Example: $\begin{vmatrix} 4 & 5 \\ -3 & 6 \end{vmatrix} = 4(6) - (5)(-3) = 39$

$16x - 10y = \sim$
$-8x + 5y = \sim$

Example 1 Evaluate each determinant.

1. $\begin{vmatrix} 8 & 6 \\ 5 & 7 \end{vmatrix}$ **26**

2. $\begin{vmatrix} -6 & -6 \\ 8 & 10 \end{vmatrix}$ **−12**

3. $\begin{vmatrix} -4 & 12 \\ 9 & 5 \end{vmatrix}$ **−128**

4. $\begin{vmatrix} 16 & -10 \\ -8 & 5 \end{vmatrix}$ **0**

1 $(8)(7) - (5)(6)$
$56 - 30 = 26$

$(16)(5) - (-8)(-10)$
$80 - 80$ 0

Step 1 Rewrite the first two columns to the right of the determinant.

$$\left|\begin{array}{rrr} 4 & -8 & 3 \\ -3 & 2 & 6 \\ -4 & 5 & 9 \end{array}\right|\begin{array}{rr} 4 & -8 \\ -3 & 2 \\ -4 & 5 \end{array}$$

Step 2 Find the products of the elements of the diagonals.

$$\left|\begin{array}{rrr} 4 & -8 & 3 \\ -3 & 2 & 6 \\ -4 & 5 & 9 \end{array}\right|\begin{array}{rr} 4 & -8 \\ -3 & 2 \\ -4 & 5 \end{array} \qquad \left|\begin{array}{rrr} 4 & -8 & 3 \\ -3 & 2 & 6 \\ -4 & 5 & 9 \end{array}\right|\begin{array}{rr} 4 & -8 \\ -3 & 2 \\ -4 & 5 \end{array}$$

$4(2)(9) = 72$

$-8(6)(-4) = 192$

$3(-3)(5) = -45$

$-4(2)(3) = -24$

$5(6)(4) = 120$

$9(-3)(-8) = 216$

Step 3 Find the sum of each group.

$72 + 192 + (-45) = 219$

$-24 + 120 + 216 = 312$

Step 4 Subtract the sum of the second group from the sum of the first group.

$219 - 312 = -93$

The value of the determinant is -93.

Example 2 Evaluate each determinant using diagonals.

5. $\begin{vmatrix} 3 & -2 & 2 \\ -4 & 2 & -5 \\ -3 & 1 & 4 \end{vmatrix}$ **−19**

5

$$\begin{vmatrix} 3 & -2 & 2 \\ -4 & 2 & -5 \\ -3 & 1 & 4 \end{vmatrix} \begin{matrix} 3 & -2 \\ -4 & 2 \\ -3 & 1 \end{matrix}$$

$3 \times 2 \times 4 = 24$

$-2 \times -5 \times -3 = -30$

$= -8$

-14

$2 \times 2 \times -3 = -12$

$= -15$

$= 32$

5

Example 2 **Evaluate each determinant using diagonals.**

5. $\begin{vmatrix} 3 & -2 & 2 \\ -4 & 2 & -5 \\ -3 & 1 & 4 \end{vmatrix}$ **−19**

6. $\begin{vmatrix} 2 & -3 & 5 \\ -4 & 6 & -2 \\ 4 & -1 & -6 \end{vmatrix}$ **−80**

7. $\begin{vmatrix} 8 & 4 & 0 \\ -2 & -6 & -1 \\ 5 & -3 & 6 \end{vmatrix}$ **−284**

8. $\begin{vmatrix} -5 & -3 & 4 \\ -2 & -4 & -3 \\ 8 & -2 & 4 \end{vmatrix}$ **302**

9. $\begin{vmatrix} 8 & 3 & 4 \\ 2 & 4 & 2 \\ 1 & 6 & 5 \end{vmatrix}$ **72**

10. $\begin{vmatrix} -4 & 3 & 0 \\ 1 & 5 & -2 \\ -1 & -8 & -3 \end{vmatrix}$ **139**

11. $\begin{vmatrix} 2 & -6 & -3 \\ 7 & 9 & -4 \\ -6 & 4 & 9 \end{vmatrix}$ **182**

12. $\begin{vmatrix} -5 & -6 & 7 \\ 4 & 0 & 5 \\ -3 & 8 & 2 \end{vmatrix}$ **562**

Examples 1–2 **Evaluate each determinant.**

26. $\begin{vmatrix} -7 & 12 \\ 5 & 6 \end{vmatrix}$ **−102**

27. $\begin{vmatrix} -8 & -9 \\ 11 & 12 \end{vmatrix}$ **3**

28. $\begin{vmatrix} -5 & 8 \\ -6 & -7 \end{vmatrix}$ **83**

29. $\begin{vmatrix} 3 & 5 & -2 \\ -1 & -4 & 6 \\ -6 & -2 & 5 \end{vmatrix}$ **−13 5**

30. $\begin{vmatrix} 2 & 0 & -6 \\ -3 & -4 & -5 \\ -2 & 5 & 8 \end{vmatrix}$ **124**

31. $\begin{vmatrix} -5 & -1 & -2 \\ 1 & 8 & 4 \\ 0 & -6 & 9 \end{vmatrix}$ **−459**

32. $\begin{vmatrix} 6 & -3 & -5 \\ 0 & -7 & 0 \\ 3 & -6 & -4 \end{vmatrix}$ **63**

33. $\begin{vmatrix} -8 & -3 & -9 \\ 0 & 0 & 0 \\ 8 & -2 & -4 \end{vmatrix}$ **0**

34. $\begin{vmatrix} 1 & 6 & 7 \\ -2 & -5 & -8 \\ 4 & 4 & 9 \end{vmatrix}$ **−13**

35. $\begin{vmatrix} 1 & -8 & -9 \\ 6 & 5 & -6 \\ -2 & -8 & 10 \end{vmatrix}$ **728**

36. $\begin{vmatrix} 5 & -5 & -5 \\ -8 & -3 & -2 \\ -2 & 4 & 6 \end{vmatrix}$ **−120**

37. $\begin{vmatrix} -4 & 1 & -2 \\ 10 & 12 & 9 \\ -6 & 0 & 13 \end{vmatrix}$ **−952**

Example 4 Solve a System of Two Equations

Solve the system by using Cramer's Rule.

$$5x - 6y = 15$$
$$3x + 4y = -29$$

$x = \dfrac{\begin{vmatrix} m & b \\ n & g \end{vmatrix}}{\lvert C \rvert}$	Cramer's Rule	$y = \dfrac{\begin{vmatrix} a & m \\ f & n \end{vmatrix}}{\lvert C \rvert}$
$= \dfrac{\begin{vmatrix} 15 & -6 \\ -29 & 4 \end{vmatrix}}{\begin{vmatrix} 5 & -6 \\ 3 & 4 \end{vmatrix}}$	Substitute values.	$= \dfrac{\begin{vmatrix} 5 & 15 \\ 3 & -29 \end{vmatrix}}{\begin{vmatrix} 5 & -6 \\ 3 & 4 \end{vmatrix}}$
$= \dfrac{15(4) - (-29)(-6)}{5(4) - (3)(-6)}$	Evaluate.	$= \dfrac{5(-29) - 3(15)}{5(4) - (3)(-6)}$
$= \dfrac{60 - 174}{20 + 18}$	Multiply.	$= \dfrac{-145 - 45}{20 + 18}$
$= -\dfrac{114}{38}$	Add and subtract.	$= -\dfrac{190}{38}$
$= -3$	Simplify.	$= -5$

Example 4 **Use Cramer's Rule to solve each system of equations.**

13. $4x - 5y = 39$ **(6, −3)**
$3x + 8y = -6$

14. $5x + 6y = 20$ **(−2, 5)**
$-3x - 7y = -29$

15. $-8a - 5b = -27$ **(4, −1)**
$7a + 6b = 22$

16. $10c - 7d = -59$ **(−8, −3)**
$6c + 5d = -63$

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$$x = \frac{\begin{vmatrix} 39 & -5 \\ -6 & 8 \end{vmatrix}}{\begin{vmatrix} 4 & -5 \\ 3 & 8 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} 4 & 39 \\ 3 & -6 \end{vmatrix}}{\begin{vmatrix} 4 & -5 \\ 3 & 8 \end{vmatrix}}$$

$$x = \frac{312 - 30}{32 - (-15)} = \frac{282}{47} = 6$$

Example 5 Solve a System of Three Equations

Solve the system by using Cramer's Rule.

$$4x + 5y - 6z = -14$$
$$3x - 2y + 7z = 47$$
$$7x - 6y - 8z = 15$$

$$x = \frac{\begin{vmatrix} m & b & c \\ n & g & h \\ p & k & \ell \end{vmatrix}}{|C|} \qquad y = \frac{\begin{vmatrix} a & m & c \\ f & n & h \\ j & p & \ell \end{vmatrix}}{|C|} \qquad z = \frac{\begin{vmatrix} a & b & m \\ f & g & n \\ j & k & p \end{vmatrix}}{|C|}$$

$$= \frac{\begin{vmatrix} -14 & 5 & -6 \\ 47 & -2 & 7 \\ 15 & -6 & -8 \end{vmatrix}}{\begin{vmatrix} 4 & 5 & -6 \\ 3 & -2 & 7 \\ 7 & -6 & -8 \end{vmatrix}} \qquad = \frac{\begin{vmatrix} 4 & -14 & -6 \\ 3 & 47 & 7 \\ 7 & 15 & -8 \end{vmatrix}}{\begin{vmatrix} 4 & 5 & -6 \\ 3 & -2 & 7 \\ 7 & -6 & -8 \end{vmatrix}} \qquad = \frac{\begin{vmatrix} 4 & 5 & -14 \\ 3 & -2 & 47 \\ 7 & -6 & 15 \end{vmatrix}}{\begin{vmatrix} 4 & 5 & -6 \\ 3 & -2 & 7 \\ 7 & -6 & -8 \end{vmatrix}}$$

$$= \frac{3105}{621} \text{ or } 5 \qquad = -\frac{1242}{621} \text{ or } -2 \qquad = \frac{2484}{621} \text{ or } 4$$

The solution of the system is (5, −2, 4).

Use Cramer's Rule to solve each system of equations.

19. ~~$-3x - 5y + 10z = -4$~~ ~~$\left(\frac{66}{7}, -\frac{116}{7}, -\frac{41}{7}\right)$~~
~~$3x - 5y - 3z = 91$~~
$6x + 8y - 7z = -35$

21. $-9x + 5y + 3z = 50$ **(−4, −2, 8)**
$7x + 8y - 2z = -60$
$-5x + 7y + 5z = 46$

23. $9a + 7b = -30$ **(−1, −3, 7)**
$8b + 5c = 11$
$-3a + 10c = 73$

25. ~~$x + y + z = 12$~~ ~~**(4, 0, 8)**~~
~~$6x - 2y - z = 16$~~
$3x + 4y + 2z = 28$

21

$$x = \frac{\left|\begin{array}{ccc|cc} 50 & 5 & 3 & 50 & 5 \\ -60 & 8 & -2 & -60 & 8 \\ 46 & 7 & 5 & 46 & 7 \end{array}\right|}{\left|\begin{array}{ccc|cc} -9 & 5 & 3 & -9 & 5 \\ 7 & 8 & -2 & 7 & 8 \\ -5 & 7 & 5 & -5 & 7 \end{array}\right|}$$

$2000 - 460 - 1260 = 280$

$1104 - 700 - 1500 = -1096$

$\boxed{-816}$

$-360 + 50 + 147 = 163$??

$-120 + 126 + 175 = 181$

$181 - 163 = 18$

I got sloppy and got it wrong! UGH!!

Examples 4–5 **Use Cramer's Rule to solve each system of equations.**

39. $6x - 5y = 73$ **(8, −5)**
$-7x + 3y = -71$

40. $10a - 3b = -34$ **(−4, −2)**
$3a + 8b = -28$

41 $-4c - 5d = -39$ **(6, 3)**
$5c + 8d = 54$

42. $-6f - 8g = -22$ **(5, −1)**
$-11f + 5g = -60$

43. $9r + 4s = -55$ **(−3, −7)**
$-5r - 3s = 36$

44. $-11u - 7v = 4$ **(−8, 12)**
$9u + 4v = -24$

45. $5x - 4y + 6z = 58$ **(4, −2, 5)**
$-4x + 6y + 3z = -13$
$6x + 3y + 7z = 53$

46. $8x - 4y + 7z = 34$ **(−3, −4, 6)**
$5x + 6y + 3z = -21$
$3x + 7y - 8z = -85$

47. **DOUGHNUTS** Mi-Ling is ordering doughnuts for a class party. The box contains 2 dozen doughnuts, some of which are plain and some of which are jelly-filled. The plain doughnuts each cost $0.50, and the jelly-filled cost $0.60. If the total cost is $12.60, use Cramer's Rule to find how many jelly-filled doughnuts Mi-Ling ordered. **6**