

3-8 Solving Systems of Equations Using Inverse Matrices

1 Identity and Inverse Matrices Recall that in real numbers, two numbers are multiplicative inverses if their product is the multiplicative identity, 1. Similarly, for matrices, the **identity matrix** is a *square matrix* that, when multiplied by another matrix, equals that same matrix. A **square matrix** is a matrix with the same number of rows and columns.

2 × 2 Identity Matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3 × 3 Identity Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Key Concept Identity Matrix for Multiplication

Words The identity matrix for multiplication I is a square matrix with 1 for every element of the main diagonal, from upper left to lower right, and 0 in all other positions. For any square matrix A of the same dimension as I , $A \cdot I = I \cdot A = A$.

Symbols If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Two $n \times n$ matrices are **inverses** of each other if their product is the identity matrix. If matrix A has an inverse symbolized by A^{-1} , then $A \cdot A^{-1} = A^{-1} \cdot A = I$.

Determine whether the matrices in each pair are inverses.

a. $A = \begin{bmatrix} -4 & 2 \\ -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ \frac{1}{2} & -1 \end{bmatrix}$

If A and B are inverses, then $A \cdot B = B \cdot A = I$.

$$A \cdot B = \begin{bmatrix} -4 & 2 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ \frac{1}{2} & -1 \end{bmatrix} \quad \text{Write an equation.}$$

$$= \begin{bmatrix} -1+1 & 2-2 \\ -\frac{1}{2}+\frac{1}{2} & 1-1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{Matrix multiplication}$$

Since $A \cdot B \neq I$, they are not inverses.

b. $F = \begin{bmatrix} 3 & -5 \\ -2 & 6 \end{bmatrix}$ and $G = \begin{bmatrix} \frac{3}{4} & \frac{5}{8} \\ \frac{1}{4} & \frac{3}{8} \end{bmatrix}$

If F and G are inverses, then $F \cdot G = G \cdot F = I$.

$$F \cdot G = \begin{bmatrix} 3 & -5 \\ -2 & 6 \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{4} & \frac{5}{8} \\ \frac{1}{4} & \frac{3}{8} \end{bmatrix} \quad \text{Write an equation.}$$

$$= \begin{bmatrix} \frac{9}{4} - \frac{5}{4} & \frac{15}{8} - \frac{15}{8} \\ -\frac{6}{4} + \frac{6}{4} & -\frac{10}{8} + \frac{18}{8} \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{Matrix multiplication}$$

$$G \cdot F = \begin{bmatrix} \frac{3}{4} & \frac{5}{8} \\ \frac{1}{4} & \frac{3}{8} \end{bmatrix} \cdot \begin{bmatrix} 3 & -5 \\ -2 & 6 \end{bmatrix} \quad \text{Write an equation.}$$

$$= \begin{bmatrix} \frac{9}{4} - \frac{10}{8} & -\frac{15}{4} + \frac{30}{8} \\ \frac{3}{4} - \frac{6}{8} & -\frac{5}{4} + \frac{18}{8} \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{Matrix multiplication}$$

Since $F \cdot G = G \cdot F = I$, F and G are inverses.

Example 1

Determine whether the matrices in each pair are inverses.

1. $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ **no**

2. $C = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}, D = \begin{bmatrix} 2 & 1 \\ 5 & -3 \end{bmatrix}$ **no**

3. $F = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, G = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$ **yes**

4. $H = \begin{bmatrix} -3 & -1 \\ -4 & -2 \end{bmatrix}, J = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$ **no**

Handwritten work for problem 3:

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$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$

$\begin{bmatrix} (-1)(-1) + (1)(0) & (-1)(-1) + (1)(-1) \\ (0)(-1) + (-1)(0) & (0)(-1) + (-1)(-1) \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The handwritten work shows the calculation of the product of matrices F and G. The result is the identity matrix, confirming they are inverses. The calculations are: $(-1)(-1) + (1)(0) = 1$, $(-1)(-1) + (1)(-1) = 0$, $(0)(-1) + (-1)(0) = 0$, and $(0)(-1) + (-1)(-1) = 1$.

$$\begin{aligned}
 & 2) \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & -3 \end{bmatrix} \\
 & = \begin{bmatrix} (2)(2) + (1)(5) & (2)(1) + (1)(-3) \\ (5)(2) + (3)(5) & (5)(1) + (3)(-3) \end{bmatrix} \\
 & = \begin{bmatrix} 4+5 & 2+-3 \\ 10+15 & 5+-15 \end{bmatrix} = \begin{bmatrix} 9 & -1 \\ 25 & -10 \end{bmatrix}
 \end{aligned}$$

THE TTY ^{Abelian}
30

like



Armando Antelekelian

KeyConcept Inverse of a 2×2 Matrix

The inverse of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$.

Find the inverse of each matrix, if it exists.

a. $P = \begin{bmatrix} 7 & -5 \\ 2 & -1 \end{bmatrix}$

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Example 2 Find the inverse of each matrix, if it exists. **5–8. See margin.**

5. $\begin{bmatrix} 6 & -3 \\ -1 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 2 & -4 \\ -3 & 0 \end{bmatrix}$

7. $\begin{bmatrix} -3 & 0 \\ 5 & 2 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$

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$(6 \times 0) - (-3 \times -1)$

$-\frac{1}{3}$

$$\begin{bmatrix} 0 & 3 \\ 1 & 6 \end{bmatrix}$$

5. $\begin{bmatrix} 0 & -1 \\ -\frac{1}{3} & -2 \end{bmatrix}$

6. $\begin{bmatrix} 0 & -\frac{1}{3} \\ -\frac{1}{4} & -\frac{1}{6} \end{bmatrix}$

7. $\begin{bmatrix} -\frac{1}{3} & 0 \\ \frac{5}{6} & \frac{1}{2} \end{bmatrix}$

8. does not exist

2 Matrix Equations Matrices can be used to represent and solve systems of equations. You can write a **matrix equation** to solve the system of equations below.

$$\begin{cases} x + 2y = 9 \\ 3x - 6y = 3 \end{cases} \rightarrow \begin{bmatrix} x + 2y \\ 3x - 6y \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$$

Write the left side of the matrix equation as the product of the coefficient matrix and the variable matrix. Write the right side as a constant matrix.

$$\begin{matrix} A & \cdot & X & = & B \\ \begin{bmatrix} 1 & 2 \\ 3 & -6 \end{bmatrix} & \cdot & \begin{bmatrix} x \\ y \end{bmatrix} & = & \begin{bmatrix} 9 \\ 3 \end{bmatrix} \end{matrix}$$

coefficient matrix

variable matrix
only the variables
of a system

constant matrix
only the constants
of a system

Then solve the matrix equation in the same way that you would solve any other equation.

$$ax = b$$

Write the equation.

$$AX = B$$

$$\left(\frac{1}{a}\right)ax = \left(\frac{1}{a}\right)b$$

Multiply each side by the inverse of the coefficient, if it exists.

$$A^{-1}AX = A^{-1}B$$

$$1x = \frac{b}{a}$$

$$\left(\frac{1}{a}\right)a = 1, A^{-1}A = I$$

$$IX = A^{-1}B$$

$$x = \frac{b}{a}$$

$$1x = x, IX = X$$

$$X = A^{-1}B$$

KeyConcept Inverse of a 2×2 Matrix

The inverse of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$.

KeyConcept Inverse of a 2×2 Matrix

The inverse of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$.

Example 3 Use a matrix equation to solve each system of equations.

9. $-2x + y = 9$ **$(-2, 5)$**
 $x + y = 3$

10. $4x - 2y = 22$ **$(4, -3)$**
 $6x + 9y = -3$

11. $-2x + y = -4$ **$(1, -2)$**
 $3x + y = 1$

12. **MONEY** Kevin had 25 quarters and dimes. The total value of all the coins was \$4.
How many quarters and dimes did Kevin have? **10 quarters and 15 dimes**

Example 1

Determine whether each pair of matrices are inverses of each other.

13. $K = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, L = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ **no**

14. $M = \begin{bmatrix} 0 & 2 \\ 4 & 5 \end{bmatrix}, N = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ **no**

15. $P = \begin{bmatrix} 4 & 0 \\ 3 & 0 \end{bmatrix}, Q = \begin{bmatrix} -1 & -1 \\ \frac{2}{3} & 5 \end{bmatrix}$ **no**

16. $R = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{2} \end{bmatrix}, S = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ **no**

Example 2

Find the inverse of each matrix, if it exists. **17–25. See margin.**

17. $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

18. $\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$

19. $\begin{bmatrix} 3 & 0 \\ 5 & 1 \end{bmatrix}$

20. $\begin{bmatrix} 1 & -1 \\ -6 & -1 \end{bmatrix}$

21. $\begin{bmatrix} -5 & -4 \\ 4 & 2 \end{bmatrix}$

22. $\begin{bmatrix} -5 & 9 \\ 4 & -8 \end{bmatrix}$

23. $\begin{bmatrix} 6 & -5 \\ 4 & 9 \end{bmatrix}$

24. $\begin{bmatrix} -4 & -2 \\ 7 & 8 \end{bmatrix}$

25. $\begin{bmatrix} -6 & 8 \\ 8 & -7 \end{bmatrix}$

Example 3

26. 6 mL of the red food coloring and 19 mL of the blue food coloring

- 26. BAKING** Peggy is preparing a colored frosting for a cake. For the right shade of purple, she needs 25 milliliters of a 44% concentration food coloring. The store has a 25% red and a 50% blue concentration of food coloring. How many milliliters each of blue food coloring and red food coloring should be mixed to make the necessary amount of purple food coloring?



PERSEVERANCE Use a matrix equation to solve each system of equations.

27. $-x + y = 4$ **no solution**
 $-x + y = -4$

30. $3x + y = 3$ $\left(\frac{3}{4}, \frac{3}{4}\right)$
 $5x + 3y = 6$

33. $1.6y - 0.2x = 1$ $(-5, 0)$
 $0.4y - 0.1x = 0.5$

28. $-x + y = 3$ $(-3, 0)$
 $-2x + y = 6$

31. $y - x = 5$ **no solution**
 $2y - 2x = 8$

34. $4y - x = -2$ $(-30, -8)$
 $3y - x = 6$

29. $x + y = 4$ $(-1, 5)$
 $-4x + y = 9$

32. $4x + 2y = 6$ $(1.5, 0)$
 $6x - 3y = 9$

35. $2y - 4x = 3$ $\left(\frac{3}{4}, 3\right)$
 $4x - 3y = -6$