$f^{-1}(x)$

3.1 Derivative of a Function

$\frac{\Delta y}{\Delta x}$

DEFINITION **Derivative**

The **derivative** of the function f with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \qquad (1)$$

provided the limit exists.

Notation

D_x

There are many ways to denote the derivative of a function $y = f(x)$. Besides $f'(x)$, the most common notations are these:

y'	"y prime"	Nice and brief, but does not name the independent variable.
noun $\frac{dy}{dx}$	"$dy\ dx$" or "the derivative of y with respect to x"	Names both variables and uses d for derivative.
$\frac{df}{dx}$	"$df\ dx$" or "the derivative of f with respect to x"	Emphasizes the function's name.
verb $\frac{d}{dx}f(x)$	"$d\ dx$ of f at x" or "the derivative of f at x"	Emphasizes the idea that differentiation is an operation performed on f.

EXAMPLE 1 Applying the Definition

Differentiate (that is, find the derivative of) $f(x) = x^3$.

SOLUTION

Applying the definition, we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h} && \text{Eq. 1 with } f(x)=x^3,\ f(x+h)=(x+h)^3 \\
&= \lim_{h\to 0}\frac{(x^3+3x^2h+3xh^2+h^3)-x^3}{h} && (x+h)^3 \text{ expanded} \\
&= \lim_{h\to 0}\frac{(3x^2+3xh+h^2)h}{h} && x^3\text{s cancelled, } h \text{ factored out} \\
&= \lim_{h\to 0}(3x^2+3xh+h^2) = 3x^2.
\end{aligned}$$

Now try Exercise 1.

In Exercises 1–4, use the definition

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

to find the derivative of the given function at the indicated point.

1. $f(x) = 1/x,\ a = 2$ $-1/4$ **2.** $f(x) = x^2 + 4,\ a = 1$ 2

3. $f(x) = 3 - x^2,\ a = -1$ 2 **4.** $f(x) = x^3 + x,\ a = 0$ 1

"Evaluate the slope..."

1

$$f'(a) = \frac{-1}{a^2}$$

$$f'(2) = \frac{-1}{2^2} = -\frac{1}{4}$$

EXAMPLE 2 Applying the Alternate Definition

Differentiate $f(x) = \sqrt{x}$ using the alternate definition.

SOLUTION

At the point $x = a$,

$$
\begin{aligned}
f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
&= \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} && \text{Eq. 2 with } f(x) = \sqrt{x} \\
&= \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} && \text{Rationalize...} \\
&= \lim_{x \to a} \frac{x - a}{(x - a)(\sqrt{x} + \sqrt{a})} && \text{...the numerator.} \\
&= \lim_{x \to a} \frac{1}{\sqrt{x} + \sqrt{a}} && \text{We can now take the limit.} \\
&= \frac{1}{2\sqrt{a}}.
\end{aligned}
$$

Applying this formula to an arbitrary $x > 0$ in the domain of f identifies the derivative as the function $f'(x) = 1/(2\sqrt{x})$ with domain $(0, \infty)$. ***Now try Exercise 5.***

In Exercises 5–8, use the definition

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

to find the derivative of the given function at the indicated point.

5. $f(x) = 1/x$, $a = 2$ $-1/4$

6. $f(x) = x^2 + 4$, $a = 1$ 2

7. $f(x) = \sqrt{x+1}$, $a = 3$ $1/4$

8. $f(x) = 2x + 3$, $a = -1$ 2

9. Find $f'(x)$ if $f(x) = 3x - 12$. $f'(x) = 3$

10. Find dy/dx if $y = 7x$. $dy/dx = 7$

5 $f(x) = \frac{1}{x}$

$f(x+h) = \frac{1}{x+h}$

LCD: $(x+h)x$

$$\lim_{h \to 0} \frac{\left(\frac{1}{x+h} - \frac{1}{x}\right)}{(h)}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{h(x(x+h))} = \lim_{h \to 0} \frac{x - x - h}{h(x(x+h))}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}(x+h)x} = \lim_{h \to 0} \frac{-1}{x(x+h)}$$

$$= \frac{-1}{x^2} = \frac{-1}{2^2} = \frac{-1}{4}$$

In Exercises 5–8, use the definition

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

to find the derivative of the given function at the indicated point.

5. $f(x) = 1/x$, $a = 2$ $-1/4$
6. $f(x) = x^2 + 4$, $a = 1$ 2
7. $f(x) = \sqrt{x+1}$, $a = 3$ $1/4$
8. $f(x) = 2x + 3$, $a = -1$ 2
9. Find $f'(x)$ if $f(x) = 3x - 12$. $f'(x) = 3$
10. Find dy/dx if $y = 7x$. $dy/dx = 7$

$f(x) = x^2+4$ $\Bigg\}$ $f(1) = 1^2+4$
$= 1+4=5$

$$f'(1) = \lim_{x \to 1} \frac{x^2+4-5}{x-1}$$

$$= \lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}$$

$$= \lim_{x \to 1} x+1 = \underline{\underline{2}}$$

10. Find dy/dx if $y = 7x$. $dy/dx = 7$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$f(x) = 7x$

$f(a) = 7a$

$$\lim_{x \to a} \frac{7x - 7a}{x - a} = \lim_{x \to a} \frac{7(x-a)}{x-a}$$

$$= \lim_{x \to a} 7 = \boxed{7}$$

In Exercises 5–8, use the definition

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

to find the derivative of the given function at the indicated point.

5. $f(x) = 1/x$, $a = 2$ $-1/4$

6. $f(x) = x^2 + 4$, $a = 1$ 2

7. $f(x) = \sqrt{x+1}$, $a = 3$ $1/4$

8. $f(x) = 2x + 3$, $a = -1$ 2

9. Find $f'(x)$ if $f(x) = 3x - 12$. $f'(x) = 3$

10. Find dy/dx if $y = 7x$. $dy/dx = 7$

7) $f(x) = \sqrt{x+1}$ $\quad f(3) = \sqrt{3+1} = \sqrt{4} = 2$

$$\lim_{x \to 3} \frac{(\sqrt{x+1} - 2)(\sqrt{x+1} + 2)}{(x-3)(\sqrt{x+1} + 2)}$$

$$= \lim_{x \to 3} \frac{(\sqrt{x+1})^2 - 4}{(x-3)(\sqrt{x+1} + 2)} = \lim_{x \to 3} \frac{x+1-4}{(x-3)(\sqrt{x+1} + 2)}$$

$$= \lim_{x \to 3} \frac{x-3}{(x-3)(\sqrt{x+1} + 2)} = \lim_{x \to 3} \frac{1}{(\sqrt{x+1} + 2)}$$

$$= \frac{1}{\sqrt{3+1} + 2} = \frac{1}{2+2}$$

In Exercises 5–8, use the definition

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

HW: complete up to #12!

to find the derivative of the given function at the indicated point.

5. $f(x) = 1/x$, $a = 2$ $\quad -1/4$ **6.** $f(x) = x^2 + 4$, $a = 1$ $\quad 2$

7. $f(x) = \sqrt{x+1}$, $a = 3$ $\quad 1/4$ **8.** $f(x) = 2x + 3$, $a = -1$ $\quad 2$

9. Find $f'(x)$ if $f(x) = 3x - 12$. $\quad f'(x) = 3$

10. Find dy/dx if $y = 7x$. $\quad dy/dx = 7$

$$f'(a) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

7 $\qquad x \Rightarrow 3$

$$f(x) = \sqrt{x+1}$$

$$f(x+h) = \sqrt{x+h+1}$$

$$\lim_{h \to 0} \left(\frac{\overset{(a}{\sqrt{x+h+1}} - \overset{b)}{\sqrt{x+1}}}{h} \right) \frac{(\overset{(a}{\sqrt{x+h+1}} + \overset{b)}{\sqrt{x+1}})}{(\sqrt{x+h+1} + \sqrt{x+1})}$$

$$a^2 - b^2$$

$$\lim_{h \to 0} \frac{(\sqrt{x+h+1})^2 - (\sqrt{x+1})^2}{h(\sqrt{x+h+1} + \sqrt{x+1})}$$

$$\lim_{h \to 0} \frac{\cancel{x} + \cancel{h} \cancel{+1} \cancel{-x} \cancel{-1}}{\cancel{h}(\sqrt{x+h+1} + \sqrt{x+1})}$$

$$\lim_{h \to 0} \frac{1}{\sqrt{x+h+1} + \sqrt{x+1}}$$

$$= \frac{1}{\sqrt{x+1} + \sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}$$

let $x = 3$

$$\frac{1}{2\sqrt{3+1}} = \boxed{\frac{1}{2\sqrt{4}}}$$

$$\frac{1}{4}$$

EXAMPLE 3 GRAPHING f' from f

Graph the derivative of the function f whose graph is shown in Figure 3.3a. Discuss the behavior of f in terms of the signs and values of f'.

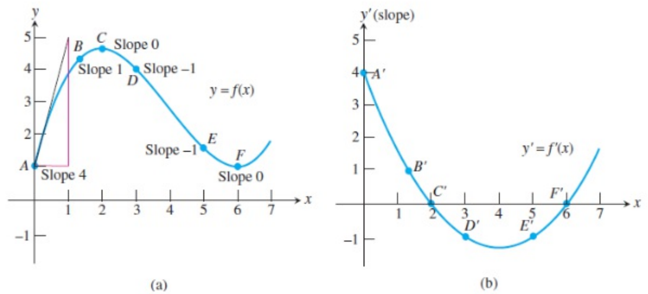
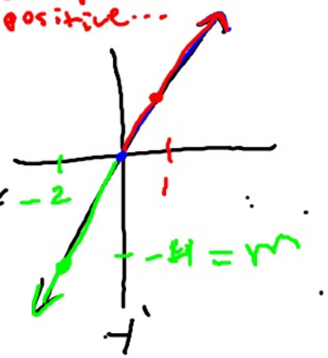


Figure 3.3 By plotting the slopes at points on the graph of $y = f(x)$, we obtain a graph of $y' = f'(x)$. The slope at point A of the graph of f in part (a) is the y-coordinate of point A' on the graph of f' in part (b), and so on. (Example 3)

$f(x) = x^2$ } $f'(x) = 2x$



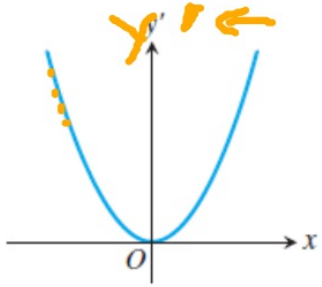
$f(x) = x^2$

x	$f(x)$
0	0
1	1
-2	4

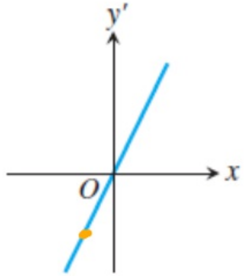
$f'(x) = 2x$

x	$f'(x)$
0	0
1	2
-2	-4

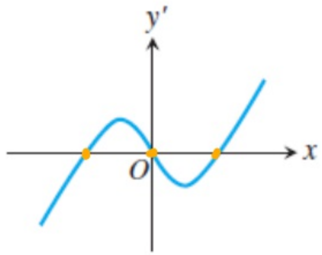
In Exercises 13–16, match the graph of the function with the graph of the derivative shown here:



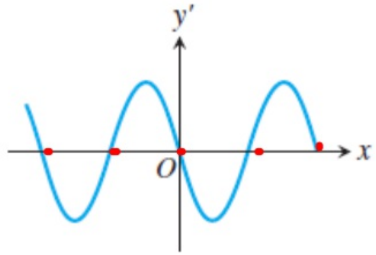
(a)



(b)

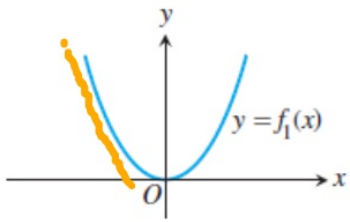


(c)

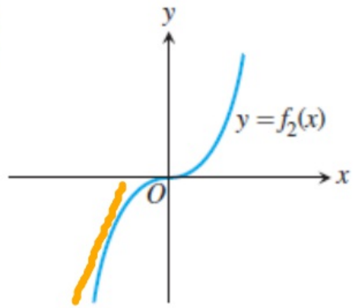


(d)

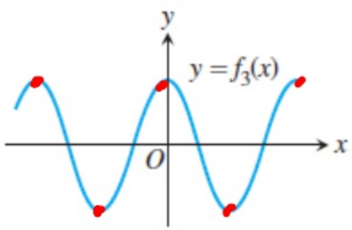
13. (b)



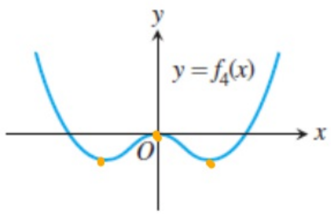
14. (a)

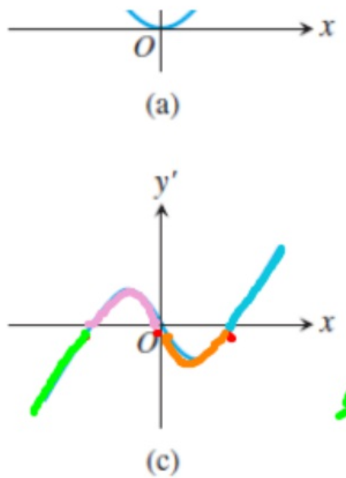


15. (d)



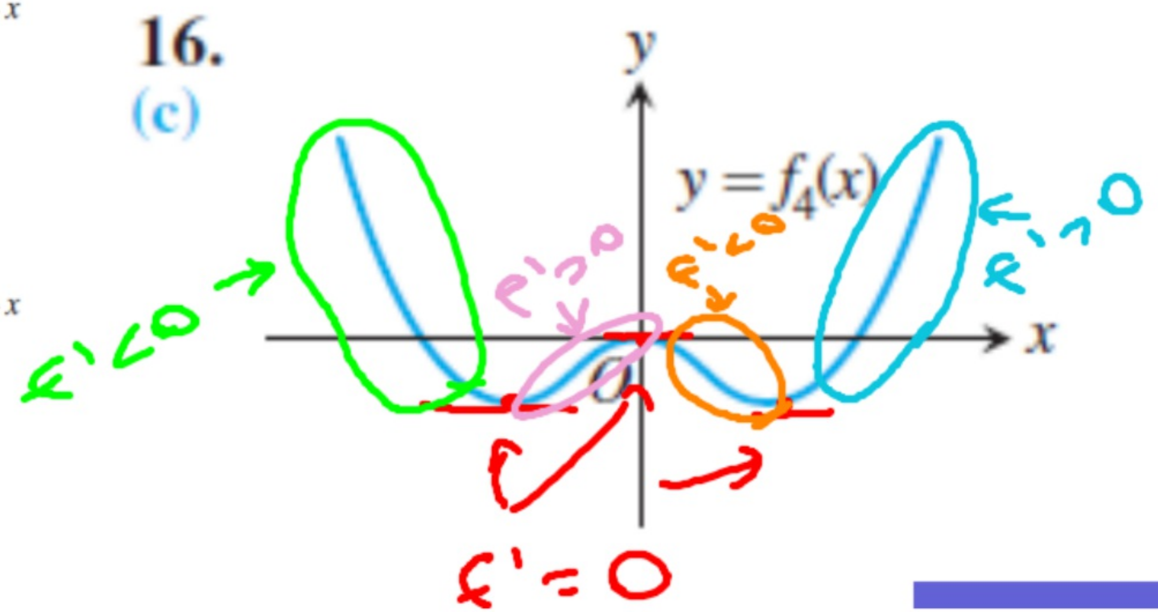
16. (c)

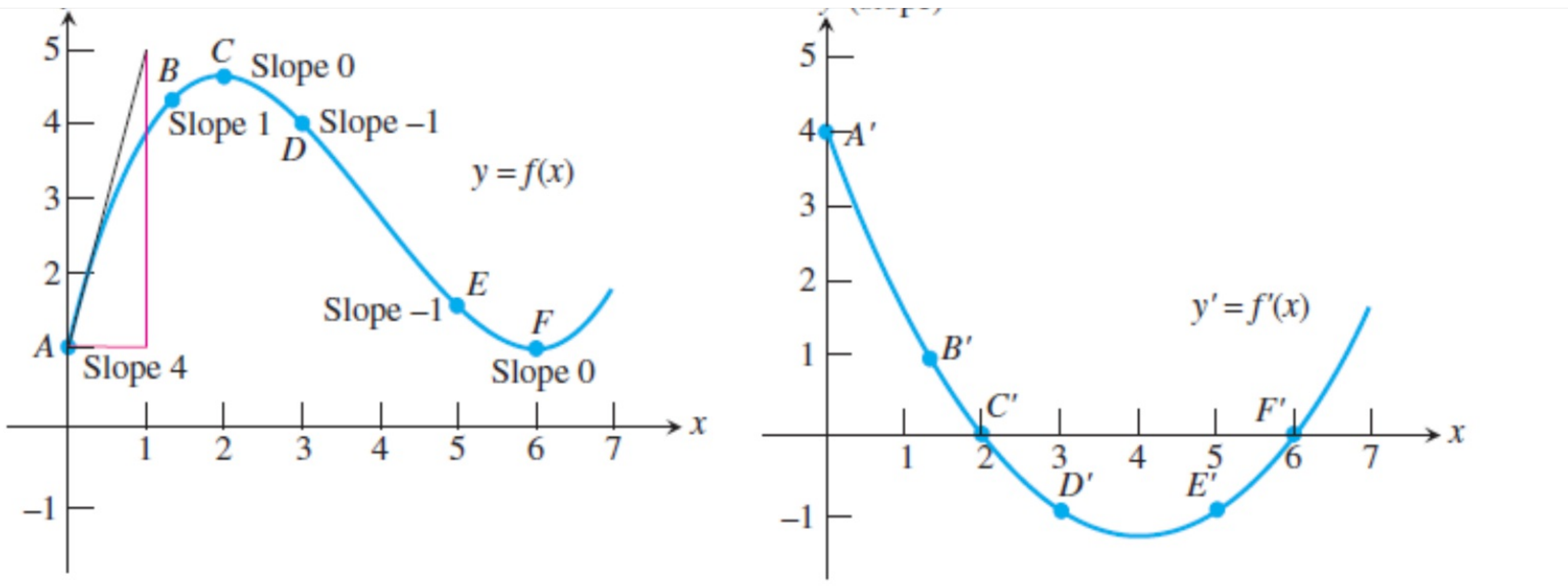




16.

(c)

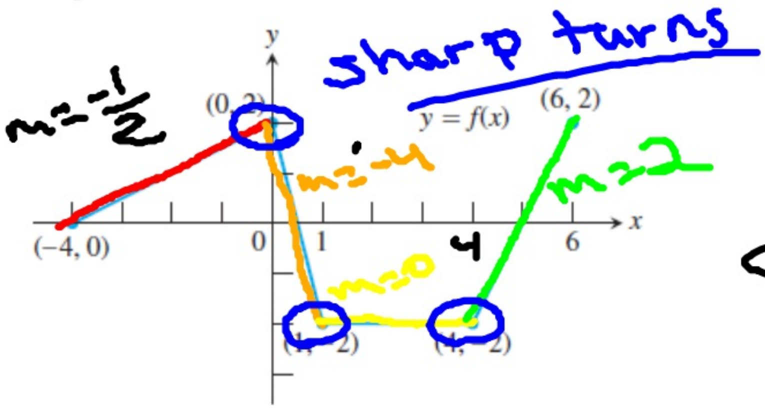




(a) (b)

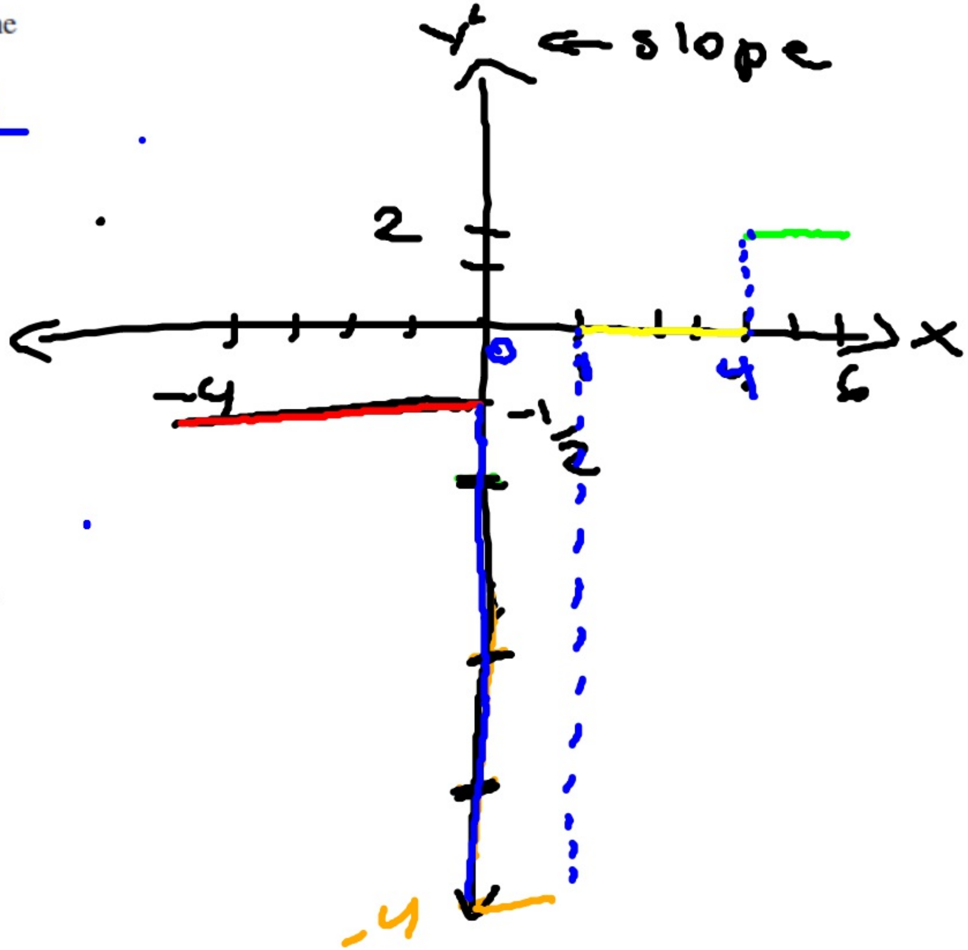
Figure 3.3 By plotting the slopes at points on the graph of $y = f(x)$, we obtain a graph of $y' = f'(x)$. The slope at point A of the graph of f in part (a) is the y-coordinate of point A' on the graph of f' in part (b), and so on. (Example 3)

26. The graph of the function $y = f(x)$ shown here is made of line segments joined end to end.



(a) Graph the function's derivative.

(b) At what values of x between $x = -4$ and $x = 6$ is the function not differentiable? $x = 0, 1, 4$



In Exercises 5–8, use the definition

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

to find the derivative of the given function at the indicated point.

5. $f(x) = 1/x$, $a = 2$ $\quad -1/4$
6. $f(x) = x^2 + 4$, $a = 1$ $\quad 2$
7. $f(x) = \sqrt{x+1}$, $a = 3$ $\quad 1/4$
8. $f(x) = 2x + 3$, $a = -1$ $\quad 2$
9. Find $f'(x)$ if $f(x) = 3x - 12$. $\quad f'(x) = 3$
10. Find dy/dx if $y = 7x$. $\quad dy/dx = 7$

LCD: 2x

5

$f(x) = \frac{1}{x}$

$f(2) = \frac{1}{2}$

$$\lim_{x\to 2} \frac{\left(\frac{1}{x} - \frac{1}{2}\right)2x}{(x-2)\,2x}$$

$$= \lim_{x\to 2} \frac{2-x}{2x(x-2)} = \lim_{x\to 2} \frac{-(x-2)}{2x(x-2)}$$

$$= \lim_{x\to 2} \frac{-1}{2x} = \frac{-1}{2(2)} = \frac{-1}{4}$$

EXAMPLE 4 Graphing f from f'

Sketch the graph of a function f that has the following properties:

i. $f(0) = 0$;

ii. the graph of f', the derivative of f, is as shown in Figure 3.4;

iii. f is continuous for all x.

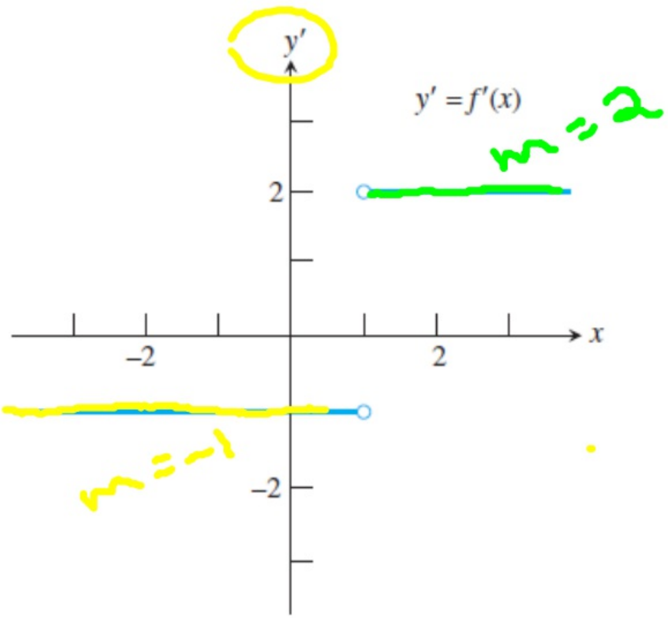


Figure 3.4 The graph of the derivative. (Example 4)

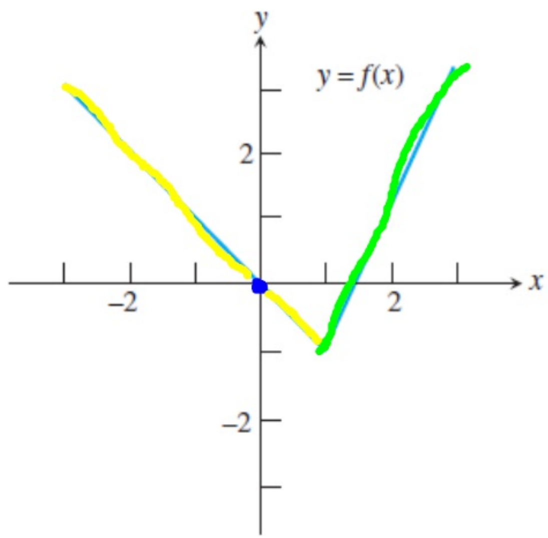
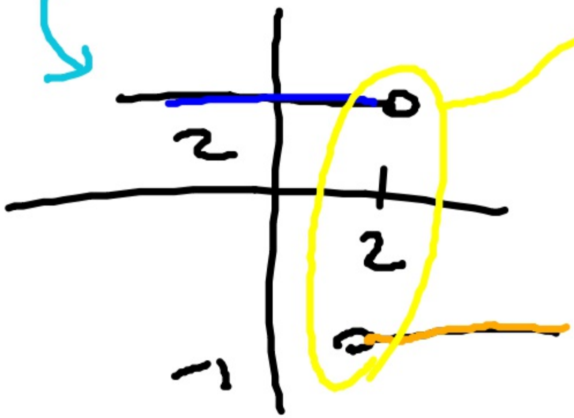
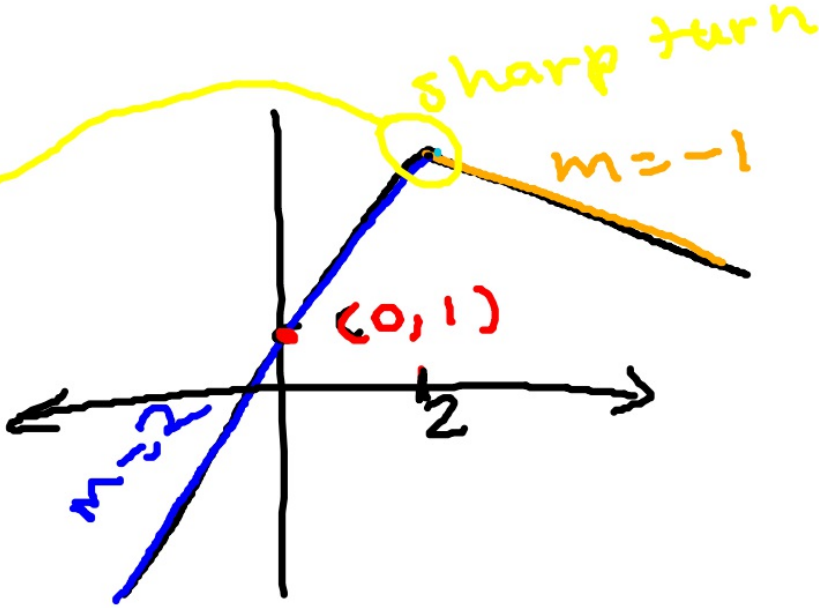


Figure 3.5 The graph of f, constructed from the graph of f' and two other conditions. (Example 4)

28. ***Graphing f from f′*** Sketch the graph of a continuous function f with $f(0) = 1$ and

$$f'(x) = \begin{cases} 2, & x < 2 \\ -1, & x > 2. \end{cases}$$





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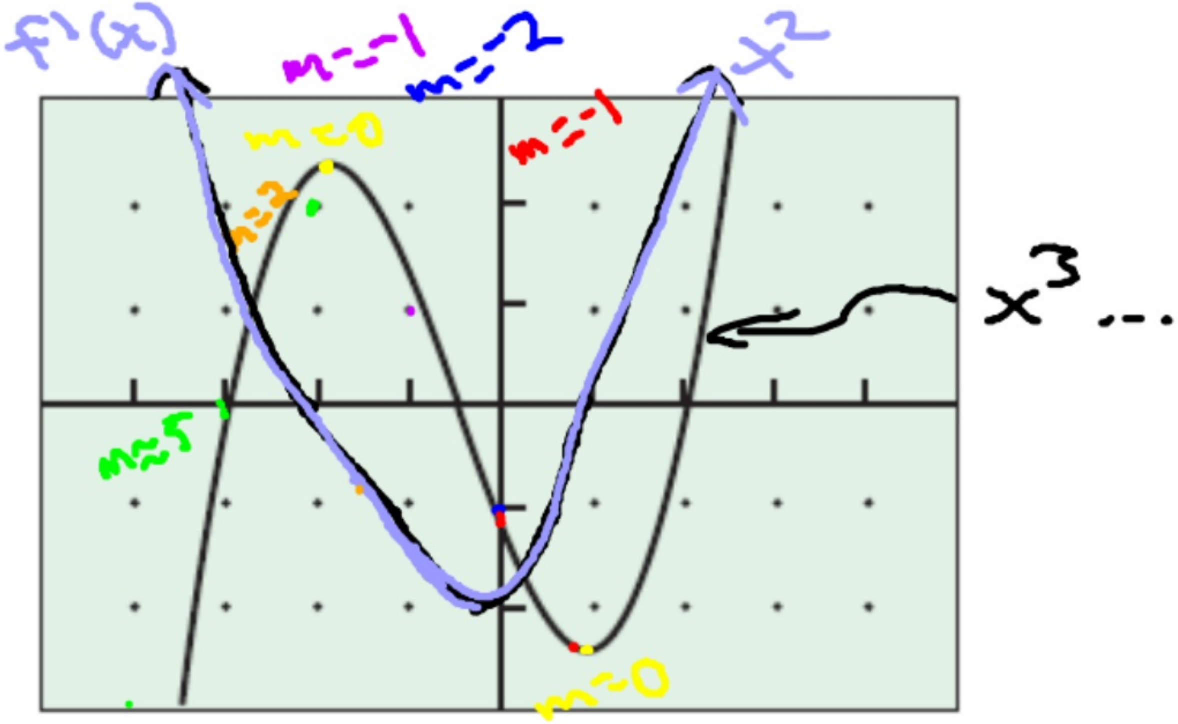
22



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