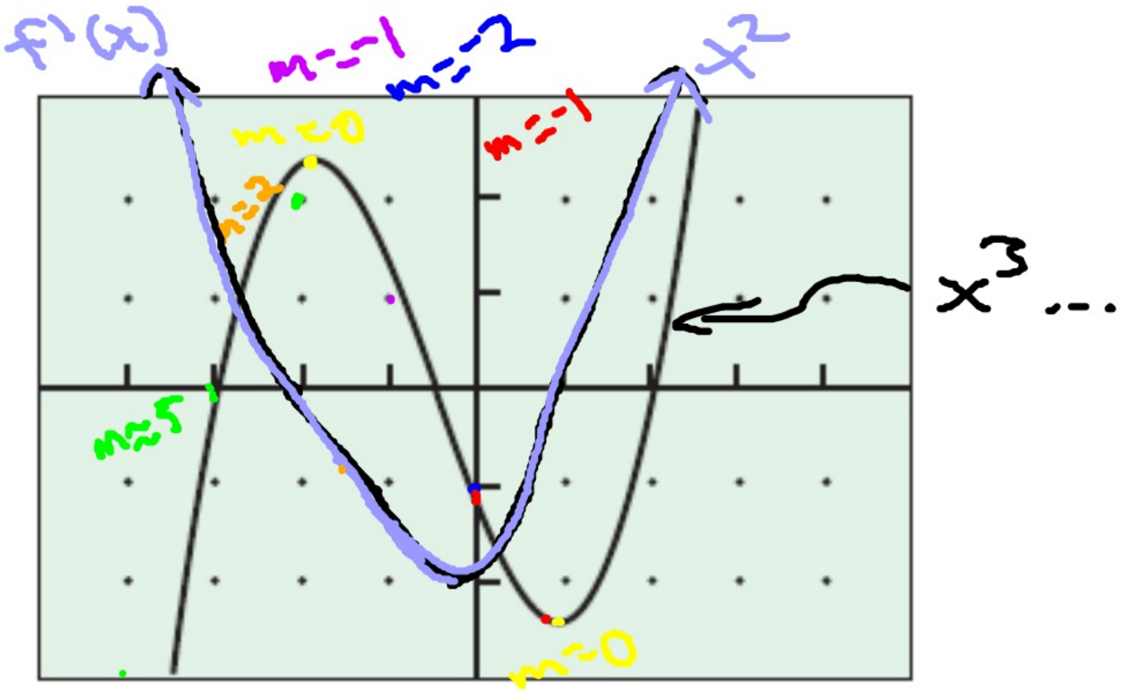
2-2



[–5, 5] by [–3, 3]

17. If $f(2) = 3$ and $f'(2) = 5$, find an equation of (**a**) the *tangent* line, and (**b**) the *normal* line to the graph of $y = f(x)$ at the point where $x = 2$.

(**a**) $y = 5x - 7$

(**b**) $y = -\frac{1}{5}x + \frac{17}{5}$

$(2, 3)$

$y = mx + b$

$m = 5$

$3 = (5)(2) + b$

$3 = 10 + b$

$b = -7$

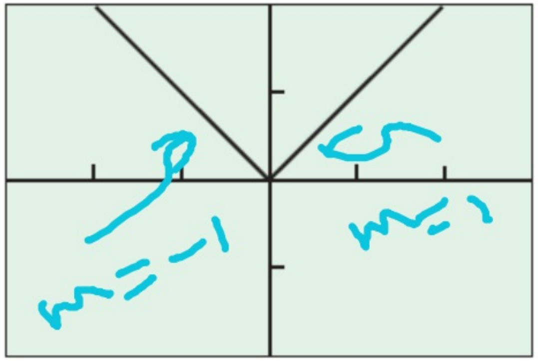
What you'll learn about

- How $f'(a)$ Might Fail to Exist
- Differentiability Implies Local Linearity
- Derivatives on a Calculator
- Differentiability Implies Continuity
- Intermediate Value Theorem for Derivatives

3.2 Differentiability

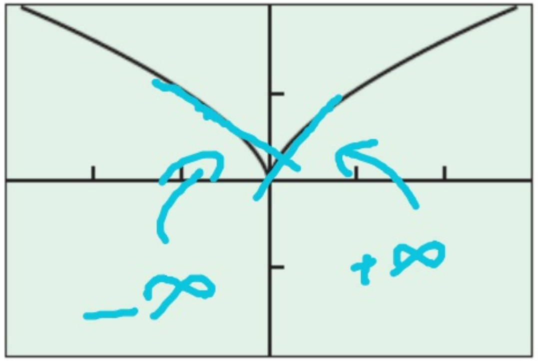
How $f'(a)$ Might Fail to Exist

1. a *corner,* where the one-sided derivatives differ; Example: $f(x) = |x|$



[–3, 3] by [–2, 2]

Figure 3.11 There is a "corner" at $x = 0$.

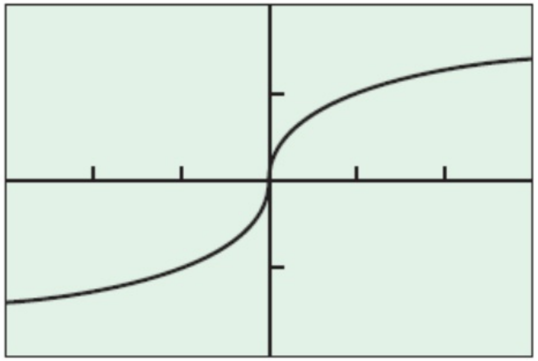


[–3, 3] by [–2, 2]

Figure 3.12 There is a "cusp" at $x = 0$.

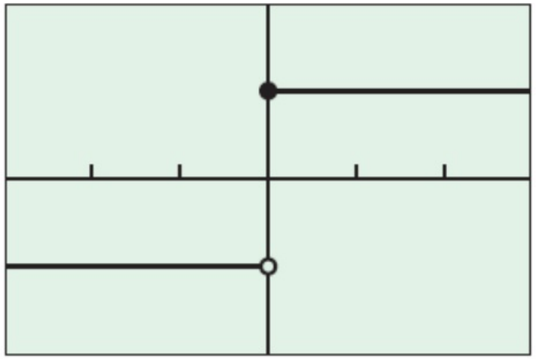
2. a *cusp,* where the slopes of the secant lines approach ∞ from one side and $-\infty$ from the other (an extreme case of a corner); Example: $f(x) = x^{2/3}$

3. a *vertical tangent,* where the slopes of the secant lines approach either ∞ or $-\infty$ from both sides (in this example, ∞); Example: $f(x) = \sqrt[3]{x}$



[–3, 3] by [–2, 2]

Figure 3.13 There is a vertical tangent line at $x = 0$.



[–3, 3] by [–2, 2]

Figure 3.14 There is a discontinuity at $x = 0$.

4. a *discontinuity* (which will cause one or both of the one-sided derivatives to be nonexistent). Example: The *Unit Step Function*

$$U(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

In this example, the left-hand derivative fails to exist:

$$\lim_{h \to 0^-} \frac{(-1) - (1)}{h} = \lim_{h \to 0^-} \frac{-2}{h} = \infty.$$

EXAMPLE 1 Finding Where a Function is not Differentiable

Find all points in the domain of $f(x) = |x - 2| + 3$ where f is not differentiable.

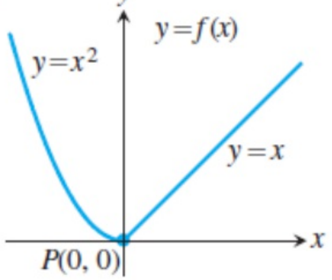
SOLUTION

Think graphically! The graph of this function is the same as that of $y = |x|$, translated 2 units to the right and 3 units up. This puts the corner at the point (2, 3), so this function is not differentiable at $x = 2$.

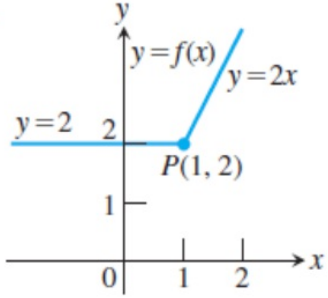
At every other point, the graph is (locally) a straight line and f has derivative $+1$ or -1 (again, just like $y = |x|$). *Now try Exercise 1.*

In Exercises 1–4, compare the right-hand and left-hand derivatives to show that the function is not differentiable at the point P. Find all points where f is not differentiable.

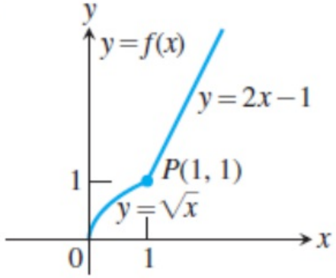
1.



2.



3.



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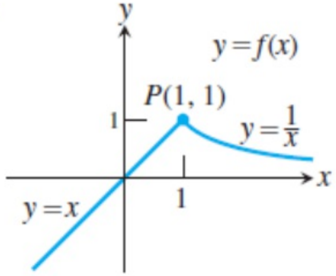


THEOREM 1 Differentiability Implies Continuity

If f has a derivative at $x = a$, then f is continuous at $x = a$.

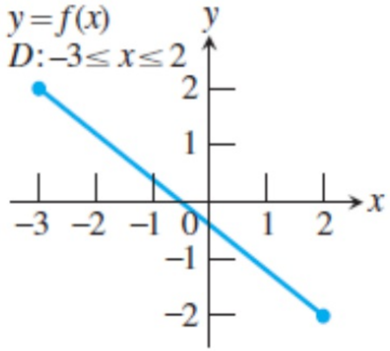
In Exercises 5–10, the graph of a function over a closed interval D is given. At what domain points does the function appear to be

(**a**) differentiable? (**b**) continuous but not differentiable?

(**c**) neither continuous nor differentiable?

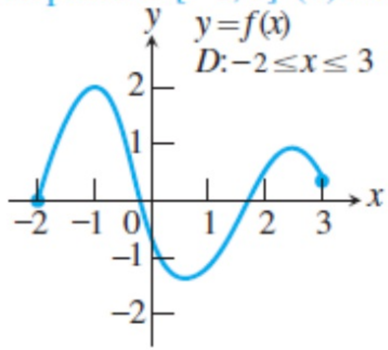
(**a**) All points in $[-3, 2]$ (**b**) None (**c**) None

5.



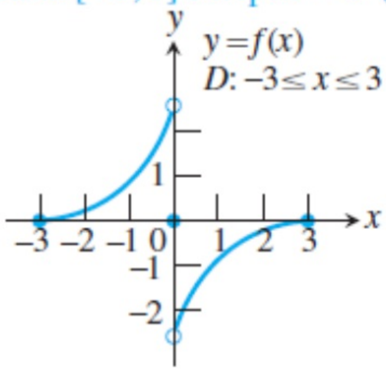
(**a**) All points in $[-2, 3]$ (**b**) None (**c**) None

6.

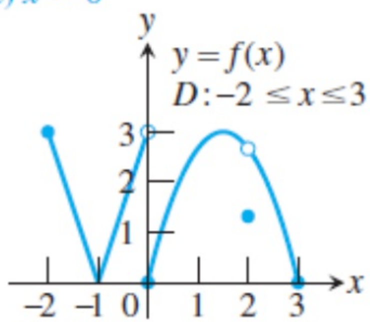


(**a**) All points in $[-3, 3]$ except $x = 0$ (**b**) None (**c**) $x = 0$

7.



8.

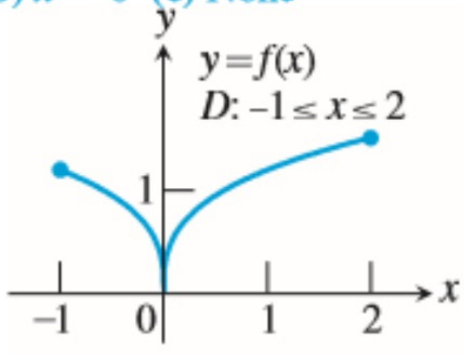


(**a**) All points in $[-2, 3]$ except $x = -1, 0, 2$
(**b**) $x = -1$ (**c**) $x = 0, x = 2$

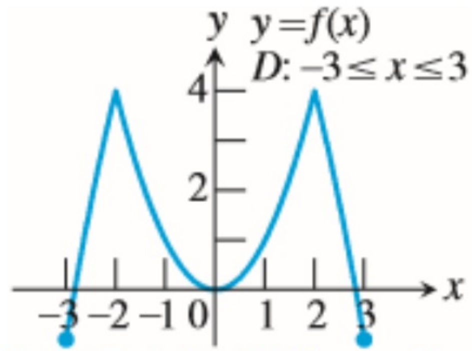
(a) All points in $[-1, 2]$ except $x = 0$

(b) $x = 0$ (c) None

9.



10.



In Exercises 11–16, the function fails to be differentiable at $x = 0$. Tell whether the problem is a corner, a cusp, a vertical tangent, or a discontinuity.

11. $y = \begin{cases} \tan^{-1} x, & x \neq 0 \\ 1, & x = 0 \end{cases}$ Discontinuity

12. $y = x^{4/5}$ Cusp

13. $y = x + \sqrt{x^2} + 2$ Corner

14. $y = 3 - \sqrt[3]{x}$ Vertical tangent

15. $y = 3x - 2|x| - 1$ Corner

16. $y = \sqrt[3]{|x|}$ Cusp

EXPLORATION 2 Looking at the Symmetric Difference Quotient Analytically

Let $f(x) = x^2$ and let $h = 0.01$.

1. Find
$$\frac{f(10+h) - f(10)}{h}.$$
How close is it to $f'(10)$?

2. Find
$$\frac{f(10+h) - f(10-h)}{2h}.$$
How close is it to $f'(10)$?

3. Repeat this comparison for $f(x) = x^3$.

Your graphing calculator can compute derivatives!

In Exercises 17–26, find the numerical derivative of the given function at the indicated point. Use $h = 0.001$. Is the function differentiable at the indicated point?

17. $f(x) = 4x - x^2,\ x = 0$ 4, yes

18. $f(x) = 4x - x^2,\ x = 3$ -2, yes

19. $f(x) = 4x - x^2,\ x = 1$ 2, yes

20. $f(x) = x^3 - 4x,\ x = 0$ -3.999999, yes

21. $f(x) = x^3 - 4x,\ x = -2$ 8.000001, yes

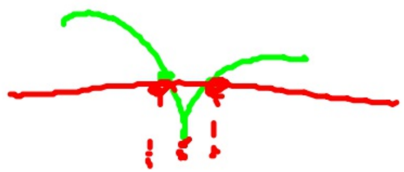
22. $f(x) = x^3 - 4x,\ x = 2$ 8.000001, yes

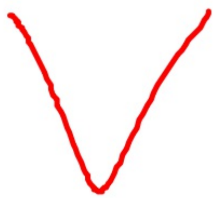
23. $f(x) = x^{2/3},\ x = 0$ 0, no

24. $f(x) = |x - 3|,\ x = 3$ 0, no

25. $f(x) = x^{2/5},\ x = 0$ 0, no

26. $f(x) = x^{4/5},\ x = 0$ 0, no





....and graph derivatives!

Group Activity In Exercises 27–30, use NDER to graph the derivative of the function. If possible, identify the derivative function by looking at the graph.

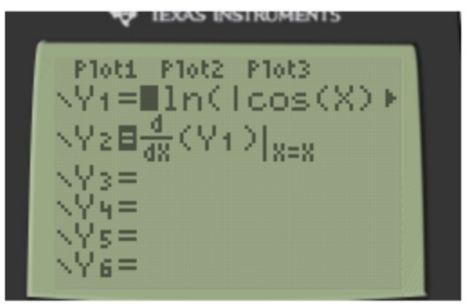
27. $y = -\cos x$

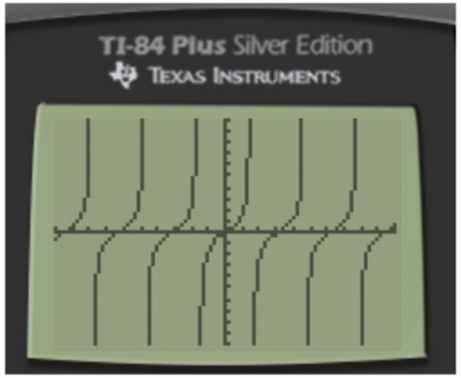
28. $y = 0.25x^4$

29. $y = \dfrac{x|x|}{2}$

30. $y = -\ln|\cos x|$

30





HW: 3.2 p.114-115 #1-35

sin |x|

THEOREM 1 Differentiability Implies Continuity

If f has a derivative at $x = a$, then f is continuous at $x = a$.

33

all values except $x = 0$

35

all values of x except $x = 3$

In Exercises 31–36, find all values of x for which the function is differentiable.

31. $f(x) = \dfrac{x^3 - 8}{x^2 - 4x - 5}$ **32.** $h(x) = \sqrt[3]{3x - 6} + 5$

33. $P(x) = \sin(|x|) - 1$ **34.** $Q(x) = 3\cos(|x|)$

34 $\mathbb{R}$

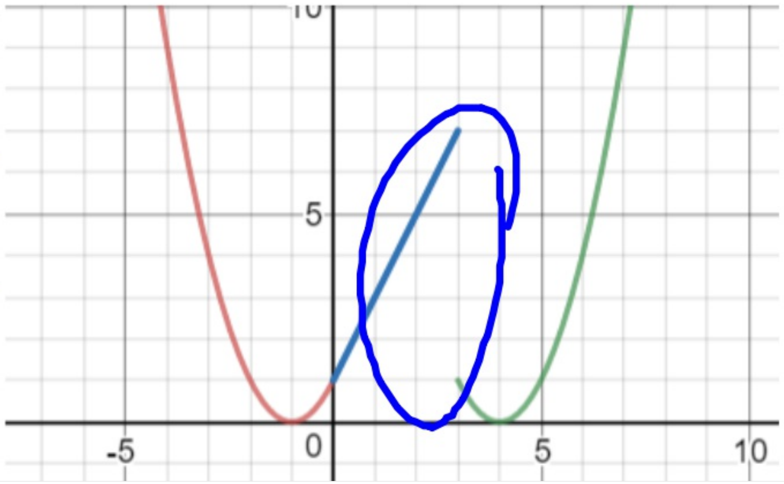
35. $g(x) = \begin{cases} (x+1)^2, & x \le 0 \\ 2x + 1, & 0 < x < 3 \\ (4 - x)^2, & x \ge 3 \end{cases}$ ← $x > 0$ and $x < 3$

35

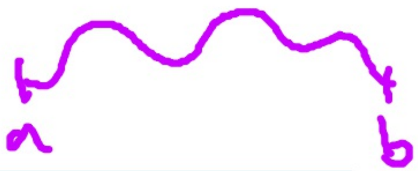
1 $(x+1)^2 \ \{x \le 0\}$

2 $2x + 1 \ \{0 < x < 3\}$

3 $(4 - x)^2 \ \{x \ge 3\}$



$f'(x)$ always exist →



THEOREM 2 Intermediate Value Theorem for Derivatives

If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.

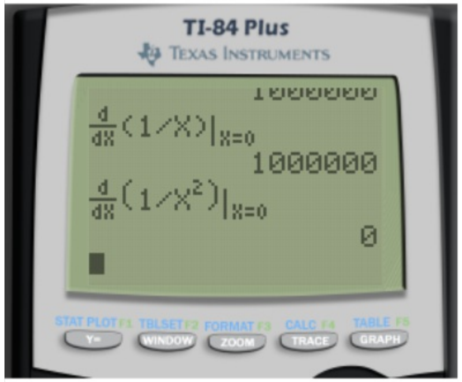
37. Show that the function

$$f(x) = \begin{cases} 0, & -1 \le x < 0 \\ 1, & 0 \le x \le 1 \end{cases}$$

is not the derivative of any function on the interval $-1 \le x \le 1$.

The function $f(x)$ does not have the intermediate value property. Choose some a in $(-1, 0)$ and b in $(0, 1)$. Then $f(a) = 0$ and $f(b) = 1$, but f does not take on any value between 0 and 1.

sharp turn.



THEOREM 2 Intermediate Value Theorem for Derivatives

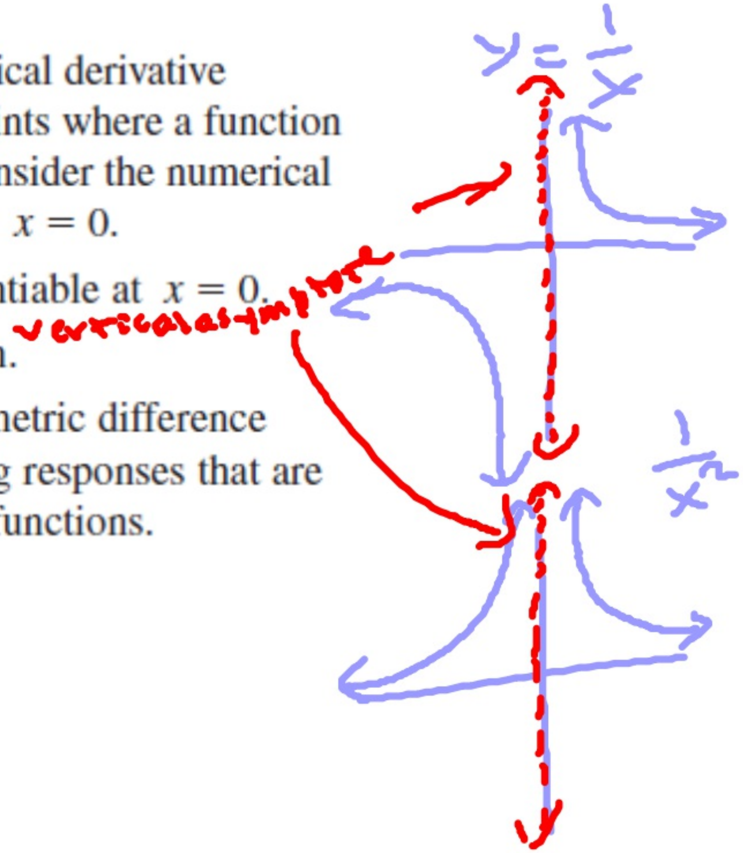
If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.

38. Writing to Learn Recall that the numerical derivative (NDER) can give meaningless values at points where a function is not differentiable. In this exercise, we consider the numerical derivatives of the functions $1/x$ and $1/x^2$ at $x = 0$.

(a) Explain why neither function is differentiable at $x = 0$.

(b) Find NDER at $x = 0$ for each function.

(c) By analyzing the definition of the symmetric difference quotient, explain why NDER returns wrong responses that are so different from each other for these two functions.



3.3 intro

$x^m \cdot x^n = x^{m+n}$

$f(x)$	$f'(x)$
$2x^{3-1}$	$6x^2$
$5x^{2-1} + 2x^{1-1}$	$10x + 2x^0$ ← scalar rule
$x^{4-1} + 7x^{3-1}$	$4x^3 + 21x^2$
$5x^7 + x^2$	$35x^6 + 2x$

$f(x) = x^n \qquad f'(x) = n \cdot x^{n-1}$ power rule

$\frac{1}{3} - 1 = -\frac{2}{3}$

$f(x)$	$f'(x)$
$f(x) = \sqrt[3]{x}$ $f(x) = x^{1/3}$	$f'(x) = \frac{1}{3}x^{-2/3}$ $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$
$f(x) = \frac{1}{x^3}$ $f(x) = x^{-3}$	$-3-1$ $f'(x) = -3x^{-4}$ $f'(x) = \frac{-3}{x^4}$
$f(x) = 5$	$f'(x) = 0$

constant rule

" Ticket out the door! "

Find the derivative $\quad -5-1 = -6$

$x^{-2/5} = \frac{1}{x^{2/5}} = \frac{1}{\sqrt[5]{x^2}}$

$$f(x) = 5x^4 + \sqrt[5]{x^3} - \frac{1}{x^5} + 3\pi$$

$x^{3/5} \qquad x^{-5} \qquad -5-1$

$$f'(x) = 20x^3 + \frac{3}{5}x^{-2/5} + 5x^{-6} + 0$$

$$= 20x^3 + \frac{3}{5\sqrt[5]{x^2}} + \frac{5}{x^6}$$