What you'll learn about
- Properties of Logarithms
- Change of Base
- Graphs of Logarithmic Functions with Base b

3.4 Properties of Logarithms

Properties of Logarithms

Let b, R, and S be positive real numbers with $b \neq 1$, and c any real number.

- **Product rule**: $\log_b (RS) = \log_b R + \log_b S$
- **Quotient rule**: $\log_b \frac{R}{S} = \log_b R - \log_b S$
- **Power rule**: $\log_b R^c = c \log_b R$

EXAMPLE 1 Proving the Product Rule for Logarithms

Prove $\log_b (RS) = \log_b R + \log_b S$.

$b^x = R$ $\quad$ $b^y = S$

$\log_b R = x$ $\quad$ $\log_b S = y$

$\log_b (RS) = \log_b b^x b^y$

$= \log_b b^{x+y} = x + y$

$= \log_b R + \log_b S$

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37. Prove the quotient rule of logarithms.

38. Prove the power rule of logarithms.

37

$b^x = R \qquad b^y = S$

$\log_b R = x \qquad \log_b S = y$

$\log_b \frac{R}{S} = \log_b \frac{b^x}{b^y} = \log_b b^{x-y}$

$= x - y = \log_b R - \log_b S$

38

$b^x = R$

$\log_b R = x$

$\log_b R^2 = \log_b R \cdot R = \log b^x b^x$

$= \log_b b^{2x} = 2x =$

$= 2 \log_b R$

EXAMPLE 3 Expanding the Logarithm of a Quotient

Assuming x is positive, use properties of logarithms to write $\ln(\sqrt{x^2+5}/x)$ as a sum or difference of logarithms or multiples of logarithms.

SOLUTION
$$\ln\frac{\sqrt{x^2+5}}{x} = \ln\frac{(x^2+5)^{1/2}}{x}$$
$$= \ln(x^2+5)^{1/2} - \ln x \quad \text{Quotient rule}$$
$$= \frac{1}{2}\ln(x^2+5) - \ln x \quad \text{Power rule}$$

$\ln 3^2 \cdot y$

Now try Exercise 9.

$9y = 3^2 \cdot y$

In Exercises 1–12, assuming x and y are positive, use properties of logarithms to write the expression as a sum or difference of logarithms or multiples of logarithms.

1. $\ln 8x$
2. $\ln 9y$
3. $\log\frac{3}{x}$
4. $\log\frac{2}{y}$
5. $\log_2 y^5$
6. $\log_2 x^{-2}$
7. $\log x^3y^2$
8. $\log xy^3$
9. $\ln\frac{x^2}{y^3}$
10. $\log 1000x^4$
11. $\log\sqrt[4]{\frac{x}{y}}$
12. $\ln\frac{\sqrt[3]{x}}{\sqrt[3]{y}}$

(1) $\ln 8 + \ln x$

$\ln 2^3$

$3\ln 2 + \ln x$

(2) $2\ln 3 + \ln y$

(7) $\log x^3 + \log y^2$

$3\log x + 2\log y$

(11) $\log\frac{\sqrt[4]{x}}{\sqrt[4]{y}}$

$= \log\frac{x^{1/4}}{y^{1/4}}$

$= \log x^{1/4} - \log y^{1/4}$

$= \frac{1}{4}\log x - \frac{1}{4}\log y$

(10) $\log 10^3 x^4$

$\log 10^3 + \log x^4$

$\log_{10} 10^3 + 4\log x$

$3 + 4\log x$

EXAMPLE 4 Condensing a Logarithmic Expression

Assuming x and y are positive, write $\ln x^5 - 2\ln(xy)$ as a single logarithm.

SOLUTION

$$
\begin{aligned}
\ln x^5 - 2\ln(xy) &= \ln x^5 - \ln(xy)^2 && \text{Power rule} \\
&= \ln x^5 - \ln(x^2y^2) \\
&= \ln \frac{x^5}{x^2y^2} && \text{Quotient rule} \\
&= \ln \frac{x^3}{y^2}
\end{aligned}
$$

Now try Exercise 13.

In Exercises 13–22, assuming x, y, and z are positive, use properties of logarithms to write the expression as a single logarithm.

13. $\log x + \log y$

14. $\log x + \log 5$

15. $\ln y - \ln 3$

16. $\ln x - \ln y$

17. $\frac{1}{3}\log x$

18. $\frac{1}{5}\log z$

19. $2\ln x + 3\ln y$

20. $4\log y - \log z$

21. $4\log(xy) - 3\log(yz)$

22. $3\ln(x^3y) + 2\ln(yz^2)$

21 $\log(xy)^4 - \log(yz)^3$

$\log \frac{x^4y^4}{y^3z^3} = \log \frac{x^4y}{z^3}$

14 $\log 5x$

17 $\log x^{1/3} = \log \sqrt[3]{x}$

19 $\ln x^2y^3$

Change-of-Base Formula for Logarithms

For positive real numbers a, b, and x with $a \neq 1$ and $b \neq 1$,

$$\log_b x = \frac{\log_a x}{\log_a b}.$$

EXAMPLE 5 Evaluating Logarithms by Changing the Base

(a) $\log_3 16 = \dfrac{\ln 16}{\ln 3} = 2.523\ldots \approx 2.52$

(b) $\log_6 10 = \dfrac{\log 10}{\log 6} = \dfrac{1}{\log 6} = 1.285\ldots \approx 1.29$

(c) $\log_{1/2} 2 = \dfrac{\ln 2}{\ln (1/2)} = \dfrac{\ln 2}{\ln 1 - \ln 2} = \dfrac{\ln 2}{-\ln 2} = -1$

Now try Exercise 23.

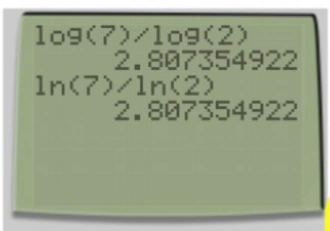
If $a = b$, then $\ln a = \ln b$ OR $\log a = \log b$.

In Exercises 23–28, use the change-of-base formula and your calculator to evaluate the logarithm.

23. $\log_2 7$ **24.** $\log_5 19$

25. $\log_8 175$ **26.** $\log_{12} 259$

27. $\log_{0.5} 12$ **28.** $\log_{0.2} 29$



23) $\log_2 7 = x$

change-of-base

$2^x = 7$

$\log 2^x = \log 7$

$\dfrac{x \cdot \log 2}{\log 2} = \dfrac{\log 7}{\log 2}$

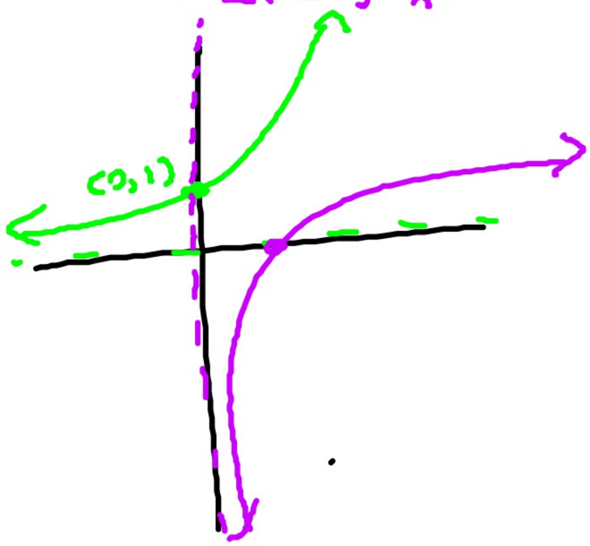
$x = \dfrac{\log 7}{\log 2}$ $\dfrac{\ln 7}{\ln 2}$

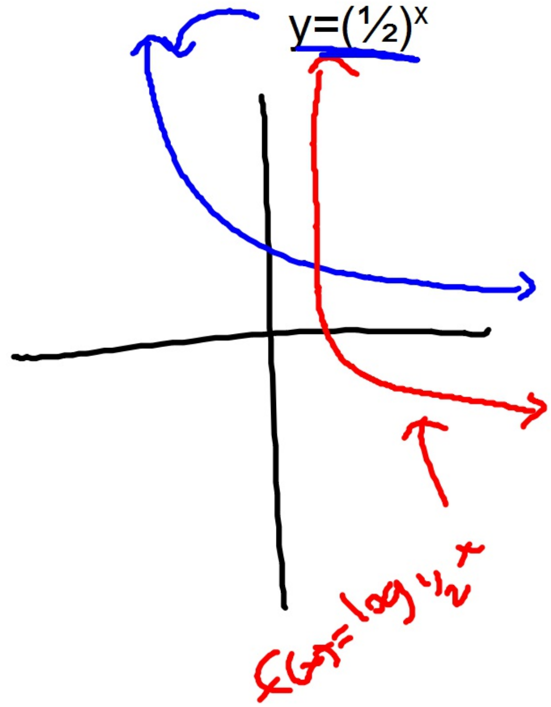
24) $\dfrac{\log 19}{\log 5} \approx ?$

We've looked at graphs of logarithms, but what happens if the base is between 0 and 1...?

2^{-x}

Ex. $y=2^x$ ← greater than 2... vs. $y=(\frac{1}{2})^x$

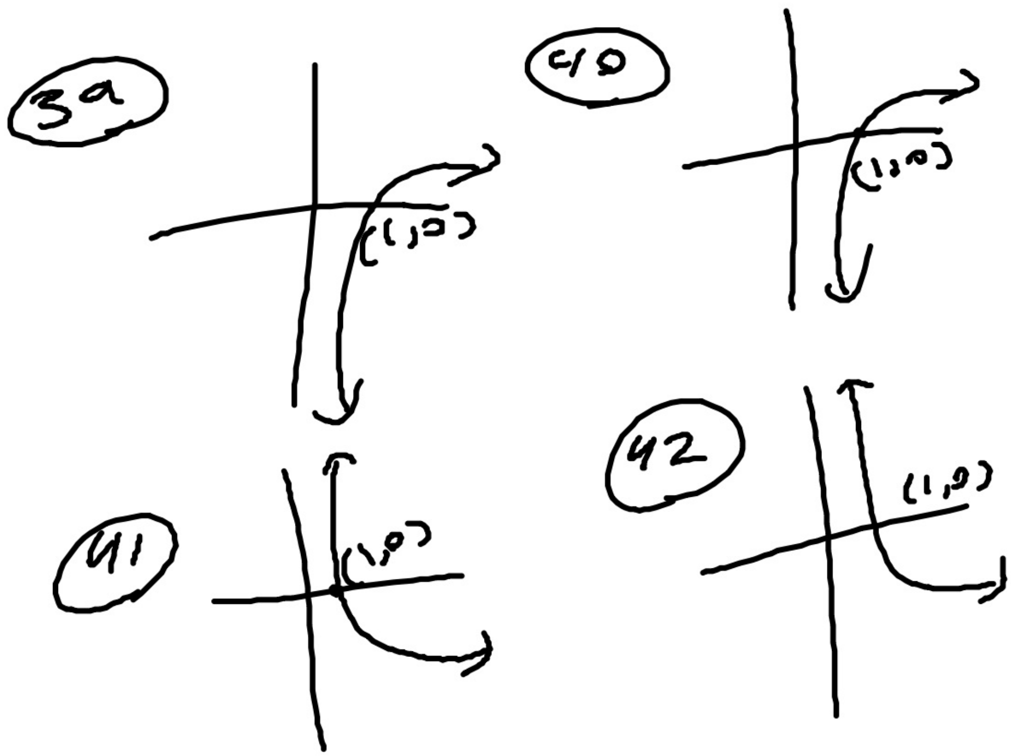




In Exercises 39–42, describe how to transform the graph of $g(x) = \ln x$ into the graph of the given function. Sketch the graph by hand and support with a grapher.

39. $f(x) = \log_4 x$ **40.** $f(x) = \log_7 x$

41. $f(x) = \log_{1/3} x$ **42.** $f(x) = \log_{1/5} x$

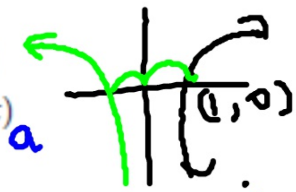


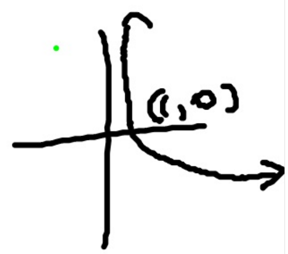
In Exercises 43–46, match the function with its graph. Identify the window dimensions, Xscl, and Yscl of the graph.

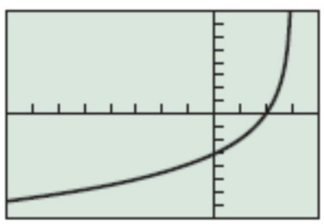
$\log_4(-(x-2))$

43. $f(x) = \log_4(2 - x)$ b **44.** $f(x) = \log_6(x - 3)$ c

45. $f(x) = \log_{0.5}(x - 2)$ d **46.** $f(x) = \log_{0.7}(3 - x)$ a







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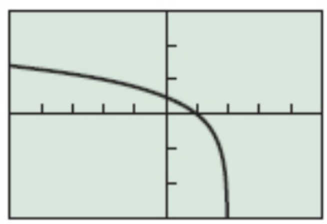

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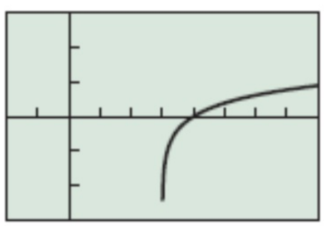

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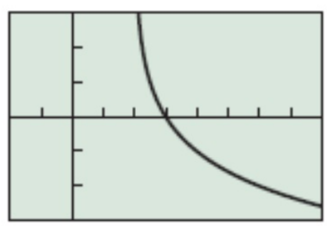

(d)