3.4 Velocity and other Rates of Change

EXAMPLE 1 Enlarging Circles

(a) Find the rate of change of the area A of a circle with respect to its radius r.

(b) Evaluate the rate of change of A at $r = 5$ and at $r = 10$.

(c) If r is measured in inches and A is measured in square inches, what units would be appropriate for dA/dr?

SOLUTION

The area of a circle is related to its radius by the equation $A = \pi r^2$.

(a) The (instantaneous) rate of change of A with respect to r is

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = \pi \cdot 2r = 2\pi r.$$

(b) At $r = 5$, the rate is 10π (about 31.4). At $r = 10$, the rate is 20π (about 62.8).

Notice that the rate of change gets bigger as r gets bigger. As can be seen in Figure 3.21, the same change in radius brings about a bigger change in area as the circles grow radially away from the center.

(c) The appropriate units for dA/dr are square inches (of area) per inch (of radius).

Now try Exercise 1.

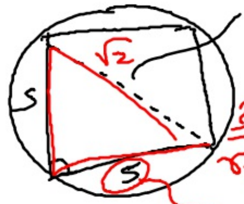
4. A square of side length s is inscribed in a circle of radius r.

(a) Write the area A of the square as a function of the radius r of the circle. $A = 2r^2$

(b) Find the (instantaneous) rate of change of the area A with respect to the radius r of the circle. $\frac{dA}{dr} = 4r$

(c) Evaluate the rate of change of A at $r = 1$ and $r = 8$. 4, 32

(d) If r is measured in inches and A is measured in square inches, what units would be appropriate for dA/dr? $\text{in}^2/\text{in.}$



$s\sqrt{2} = d$

$\frac{2}{\sqrt{2}} \cdot r = \frac{\sqrt{2}}{2} s \cdot \frac{2}{\sqrt{2}}$

$s = \frac{2}{\sqrt{2}} r$

$A = s^2 = \left(\frac{2}{\sqrt{2}} r\right)^2$

$A = \frac{4}{2} r^2 = 2r^2$

b) $\frac{d}{dr} A = \frac{d}{dr} 2r^2$

$r = 1$

$r = 8$

c) "The rate of change of A": $\frac{dA}{dr} = 4r$

...remember what "average" rate means...?

$(30-t)(30-t)$

900×200

$m = \frac{y_2 - y_1}{x_2 - x_1}$

8. At the end of 10 minutes: 8000 gallons/minute
Average over first 10 minutes: 10,000 gallons/minute

8. ***Draining a Tank*** The number of gallons of water in a tank t minutes after the tank has started to drain is $Q(t) = 200(30-t)^2$. How fast is the water running out at the end of 10 min? What is the average rate at which the water flows out during the first 10 min?

t	$Q(t)$
0	180000
10	80,000

$200(900-60t+t^2)$

$200(-60+2t) = Q'(t)$

let $t=10$;

$200(-60+2(10))$

$200(-60+20)$

$200(-40)$

$Q'(10) = -8000$

DEFINITION Speed

Speed is the absolute value of velocity.

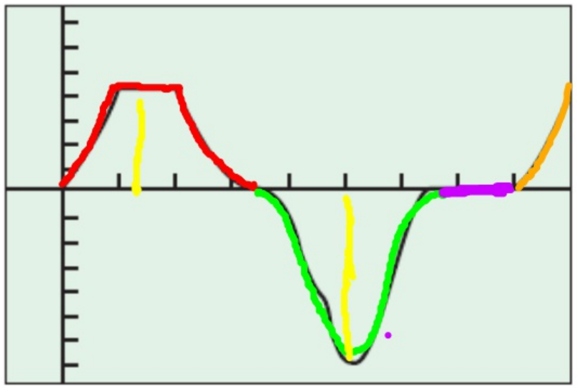
$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

EXAMPLE 3 Reading a Velocity Graph

A student walks around in front of a motion detector that records her velocity at 1-second intervals for 36 seconds. She stores the data in her graphing calculator and uses it to generate the time-velocity graph shown in Figure 3.25. Describe her motion as a function of time by reading the velocity graph. When is her *speed* a maximum?

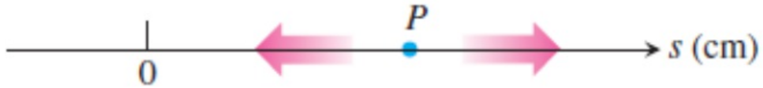
SOLUTION

The student moves forward for the first 14 seconds, moves backward for the next 12 seconds, stands still for 6 seconds, and then moves forward again. She achieves her maximum speed at $t \approx 20$, while moving backward. ***Now try Exercise 9.***

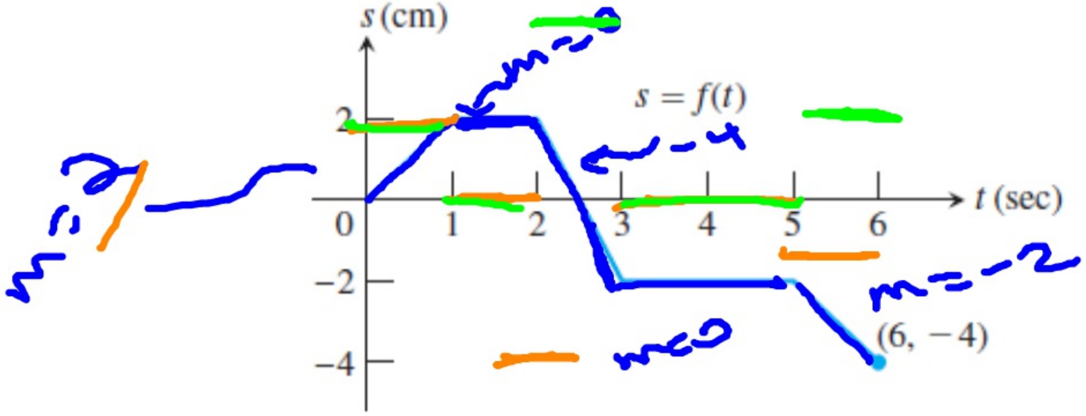


10. ***Particle Motion*** A particle P moves on the number line shown in part (a) of the accompanying figure. Part (b) shows the position of P as a function of time t.

$$\frac{\Delta y}{\Delta x} = \frac{cm}{sec}$$



(a)



(b)

(a) When is P moving to the left? moving to the right? standing still? See page 140.

(b) Graph the particle's velocity and speed (where defined).

Motion along a Line

Suppose that an object is moving along a coordinate line (say an s-axis) so that we know its position s on that line as a function of time t:

$$s = f(t).$$

The **displacement** of the object over the time interval from t to $t + \Delta t$ is

$$\Delta s = f(t + \Delta t) - f(t)$$

$\Delta x = x_2 - y_1$

(Figure 3.23) and the **average velocity** of the object over that time interval is

$$v_{\text{av}} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

DEFINITION Instantaneous Velocity

The **(instantaneous) velocity** is the derivative of the position function $s = f(t)$ with respect to time. At time t the velocity is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

Free-fall Constants (Earth)

English units: $g = 32 \dfrac{\text{ft}}{\text{sec}^2}$, $s = \dfrac{1}{2}(32)t^2 = 16t^2$ (s in feet)

Metric units: $g = 9.8 \dfrac{\text{m}}{\text{sec}^2}$, $s = \dfrac{1}{2}(9.8)t^2 = 4.9t^2$ (s in meters)

EXAMPLE 4 Modeling Vertical Motion

A dynamite blast propels a heavy rock straight up with a launch velocity of 160 ft/sec (about 109 mph) (Figure 3.26a). It reaches a height of $s = 160t - 16t^2$ ft after t seconds.

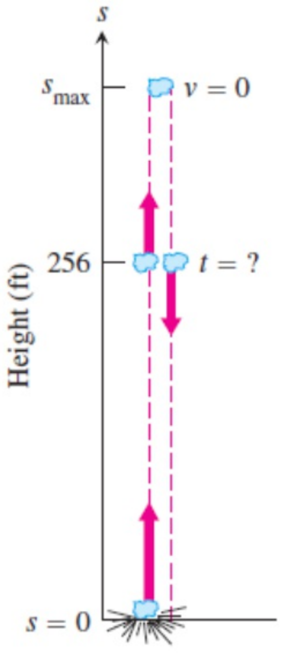
(a) How high does the rock go?

(b) What is the velocity and speed of the rock when it is 256 ft above the ground on the way up? on the way down?

(c) What is the acceleration of the rock at any time t during its flight (after the blast)?

(d) When does the rock hit the ground?

derive, set equal to zero...



(a)

(a) The instant when the rock is at its highest point is the one instant during the flight when the velocity is 0. At any time t, the velocity is

$$v = \frac{ds}{dt} = \frac{d}{dt}(160t - 16t^2) = 160 - 32t \text{ ft/sec.}$$

The velocity is zero when $160 - 32t = 0$, or at $t = 5$ sec. ← $f'(x)$

The maximum height is the height of the rock at $t = 5$ sec. That is,

$$s_{max} = s(5) = 160(5) - 16(5)^2 = 400 \text{ ft.}$$ ← $f(x)$

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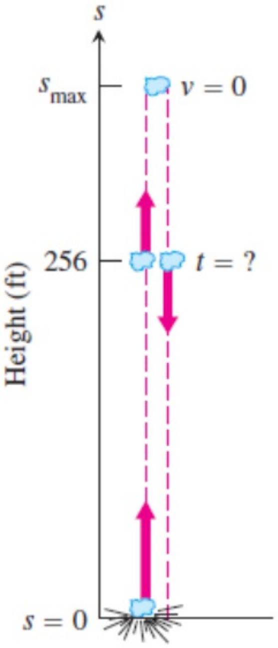

(a)

(b) To find the velocity when the height is 256 ft, we determine the two values of t for which $s(t) = 256$ ft.

$$s(t) = 160t - 16t^2 = 256$$

$$16t^2 - 160t + 256 = 0$$

$$16(t^2 - 10t + 16) = 0$$

$$(t - 2)(t - 8) = 0$$

$$t = 2 \text{ sec} \quad \text{or} \quad t = 8 \text{ sec}$$

The velocity of the rock at each of these times is

$$v(2) = 160 - 32(2) = 96 \text{ ft/sec},$$

$$v(8) = 160 - 32(8) = -96 \text{ ft/sec}.$$

At both instants, the speed of the rock is 96 ft/sec.

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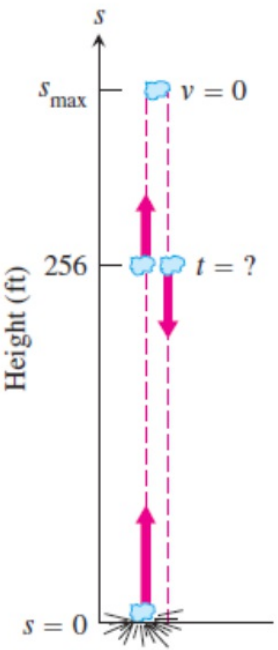


(a)

(c) At any time during its flight after the explosion, the rock's acceleration is

$$a = \frac{dv}{dt} = \frac{d}{dt}(160 - 32t) = -32 \text{ ft/sec}^2.$$

The acceleration is always downward. When the rock is rising, it is slowing down; when it is falling, it is speeding up.

(d) The rock hits the ground at the positive time for which $s = 0$. The equation $160t - 16t^2 = 0$ has two solutions: $t = 0$ and $t = 10$. The blast initiated the flight of the rock from ground level at $t = 0$. The rock returned to the ground 10 seconds later.

Now try Exercise 13.

14. ***Free Fall*** The equations for free fall near the surfaces of Mars and Jupiter (s in meters, t in seconds) are: Mars, $s = 1.86t^2$; Jupiter, $s = 11.44t^2$. How long would it take a rock falling from rest to reach a velocity of 16.6 m/sec (about 60 km/h) on each planet? Mars: $t \approx 4.462$ sec; Jupiter: $t \approx 0.726$ sec

<u>Mars</u>

$s = 1.86t^2$

$v = s' = 3.72t$

$3.72t = 16.6$

<u>Jupiter</u>

$s = 11.44t^2$

$s' = 22.88t$

$22.88t = 16.6$

22. ***Particle Motion*** A particle moves along a line so that its position at any time $t \geq 0$ is given by the function $s(t) = t^3 - 6t^2 + 8t + 2$ where s is measured in meters and t is measured in seconds.

(a) Find the instantaneous velocity at any time t.

(b) Find the acceleration of the particle at any time t.

(c) When is the particle at rest?

(d) Describe the motion of the particle. At what values of t does the particle change directions?

EXAMPLE 7 Marginal Cost and Marginal Revenue

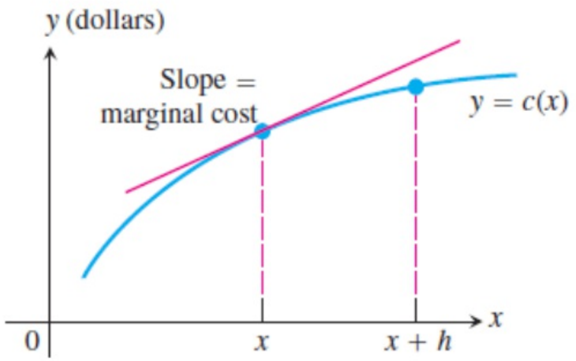
Suppose it costs

$$c(x) = x^3 - 6x^2 + 15x$$

dollars to produce x radiators when 8 to 10 radiators are produced, and that

$$r(x) = x^3 - 3x^2 + 12x$$

gives the dollar revenue from selling x radiators. Your shop currently produces 10 radiators a day. Find the marginal cost and **marginal revenue.**



SOLUTION

The marginal cost of producing one more radiator a day when 10 are being produced is $c'(10)$.

$$c'(x) = \frac{d}{dx}(x^3 - 6x^2 + 15x) = 3x^2 - 12x + 15$$

$$c'(10) = 3(100) - 12(10) + 15 = 195 \text{ dollars}$$

The marginal revenue is

$$r'(x) = \frac{d}{dx}(x^3 - 3x^2 + 12x) = 3x^2 - 6x + 12,$$

so,

$$r'(10) = 3(100) - 6(10) + 12 = 252 \text{ dollars.}$$

Now try Exercises 27 and 28.

HW: p.137-138 #1-28

28. ***Marginal Revenue*** Suppose the weekly revenue in dollars from selling x custom-made office desks is

$$r(x) = 2000\left(1 - \frac{1}{x+1}\right).$$

(a) Draw the graph of r. What values of x make sense in this problem situation?

(b) Find the marginal revenue when x desks are sold.

(c) Use the function $r'(x)$ to estimate the increase in revenue that will result from increasing sales from 5 desks a week to 6 desks a week.

(d) **Writing to Learn** Find the limit of $r'(x)$ as $x \to \infty$. How would you interpret this number?