3.5 Derivatives of Trigonometric Numbers

once upon a time, the Sandwich Theorem gave two special trigonometric limits;

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \quad \text{and} \quad \lim_{x \to 0} \frac{1-\cos(x)}{x.} = 0$$

$\frac{d}{dx}[\cos x]$?

$f(x) = \sin x$

These could be used to find the derivative of sine by using the limit definition of a derivative...

$\frac{d}{dx}[\sin x]$?

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= \lim_{x \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{x \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{x \to 0} \frac{\sin x \cos h - \sin x + \cos x \sin h}{h}$$

$$= \lim_{x \to 0} \sin x \frac{(\cos h - 1)}{h} + \lim_{x \to 0} \cos x \frac{\sin h}{h}$$

$$= \sin x(0) + \cos x(1) = \cos x$$

$$\frac{d}{dx}[\sin x] = \cos x$$

We can use the derivatives of sine and cosine, along with the Quotient Rule, to derive the other four trig functions!

$\sin^2 x + \cos^2 x = 1$

25. Assuming that $(d/dx)(\sin x) = \cos x$ and $(d/dx)(\cos x) = -\sin x$, prove each of the following.

(a) $\dfrac{d}{dx}\tan x = \sec^2 x$ (b) $\dfrac{d}{dx}\sec x = \sec x \tan x$

a) $\dfrac{d}{dx}\left[\dfrac{\sin x}{\cos x}\right] = \dfrac{\cos x[\cos x] - \sin x[-\sin x]}{\cos^2 x}$

$= \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} = \sec^2 x$

b) $\dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right) = \dfrac{\cos x(0) - (1)(-\sin x)}{\cos^2 x}$

$= \dfrac{\sin x}{\cos^2 x} = \left(\dfrac{\sin x}{\cos x}\right)\left(\dfrac{1}{\cos x}\right)$

$= (\tan x)(\sec x)$

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(a) $\dfrac{d}{dx}\tan x = \sec^2 x$

(b) $\dfrac{d}{dx}\sec x = \sec x \tan x$

26. Assuming that $(d/dx)(\sin x) = \cos x$ and $(d/dx)(\cos x) = -\sin x$, prove each of the following.

(a) $\dfrac{d}{dx}\cot x = -\csc^2 x$ **(b)** $\dfrac{d}{dx}\csc x = -\csc x \cot x$

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\csc x = -\csc x\cot x$$
$$\frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\sec x = \sec x\tan x$$
$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

In Exercises 1–10, find dy/dx. Use your grapher to support your analysis if you are unsure of your answer.

1. $y = 1 + x - \cos x$ $\quad 1 + \sin x$
2. $y = 2\sin x - \tan x$ $\quad 2\cos x - \sec^2 x$
3. $y = \frac{1}{x} + 5\sin x$ $\quad -\frac{1}{x^2} + 5\cos x$
4. $y = x\sec x$ $\quad x\sec x\tan x + \sec x$
5. $y = 4 - x^2\sin x$
6. $y = 3x + x\tan x$ $\quad 3 + x\sec^2 x + \tan x$
7. $y = \frac{4}{\cos x}$ $\quad 4\sec x\tan x$
8. $y = \frac{x}{1 + \cos x}$ $\quad \frac{1 + \cos x + x\sin x}{(1 + \cos x)^2}$
9. $y = \frac{\cot x}{1 + \cot x}$ See page 147.
10. $y = \frac{\cos x}{1 + \sin x}$ $\quad -\frac{1}{1 + \sin x}$

(6) $y = 3x + \overset{f}{x}\,\overset{g}{\tan x}$

$y' = 3 + \tan x + x\sec^2 x$

$f = x \quad g = \tan x$

$f' = 1 \quad g' = \sec^2 x$

(4) $\overset{f}{x}\,\overset{g}{\sec x}$

$= \sec x + x\sec x\tan x$

$f = x \quad g = \sec x$

$f' = 1 \quad g' = \sec x\tan x$

(7) $4\sec x$

5. $y = 4 - x^2 \sin x$

f g

$f = -x^2$ $g = \sin x$

$f' = -2x$ $g' = \cos x$

$$y = \underbrace{-2x}_{f'}(\underbrace{\sin x}_{g}) + (\underbrace{-x^2}_{f})(\underbrace{\cos x}_{g'})$$

$$y = -2x\sin x - x^2\cos x$$

2. $y = 2\sin x - \tan x$ $2\cos x - \sec^2 x$

$$y' = 2\cos x - \sec^2 x$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

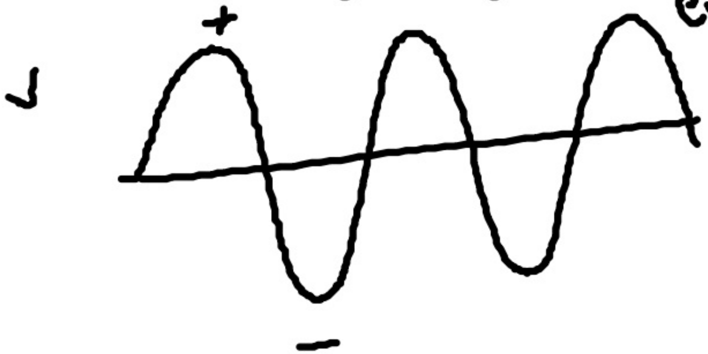
In Exercises 11 and 12, a weight hanging from a spring (see Figure 3.38) bobs up and down with position function $s = f(t)$ (s in meters, t in seconds). What are its velocity and acceleration at time t? Describe its motion.

11. $s = 5 \sin t$ **12.** $s = 7 \cos t$

11 $v = s' = 5\cos t$

$a = v' = -5\sin t$

changes direction every π



$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

$$\frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\csc x = -\csc x\cot x$$

$$\frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\sec x = \sec x\tan x$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

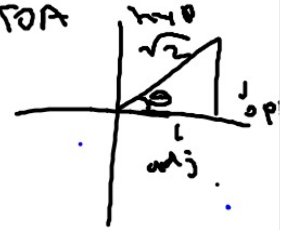
In Exercises 13–16, a body is moving in simple harmonic motion with position function $s = f(t)$ (s in meters, t in seconds).

(a) Find the body's velocity, speed, and acceleration at time t.

(b) Find the body's velocity, speed, and acceleration at time $t = \pi/4$.

(c) Describe the motion of the body.

SOHCAHTOA



13. $s = 2 + 3\sin t$ **14.** $s = 1 - 4\cos t$

15. $s = 2\sin t + 3\cos t$ **16.** $s = \cos t - 3\sin t$

a) $v = s' = -\sin t - 3\cos t$

speed $= |-\sin t - 3\cos t|$

$a = v' = -\cos t + 3\sin t$

b) $v\left(\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) - 3\cos\left(\frac{\pi}{4}\right)$

$= -\frac{1}{\sqrt{2}} - 3\left(\frac{1}{\sqrt{2}}\right) = \frac{-1}{\sqrt{2}} - \frac{3}{\sqrt{2}}$

$= \frac{-4}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

$= -2\sqrt{2}$

speed $= |-2\sqrt{2}| = 2\sqrt{2}$

$a\left(\frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) + 3\sin\left(\frac{\pi}{4}\right)$

$= -\frac{1}{\sqrt{2}} + 3\left(\frac{1}{\sqrt{2}}\right) = \frac{-1}{\sqrt{2}} + \frac{3}{\sqrt{2}} = 2\sqrt{2}$

Think of, *"The Fast and the Furious,"* movies, when the nitrous is used to get an extra burst of speed...

DEFINITION Jerk

Jerk is the derivative of acceleration. If a body's position at time t is $s(t)$, the body's jerk at time t is

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

In Exercises 17–20, a body is moving in simple harmonic motion with position function $s = f(t)$ (s in meters, t in seconds). Find the jerk at time t.

17. $s = 2\cos t$

18. $s = 1 + 2\cos t$

19. $s = \sin t - \cos t$

20. $s = 2 + 2\sin t$

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

18

$s = 1 + 2\cos t$

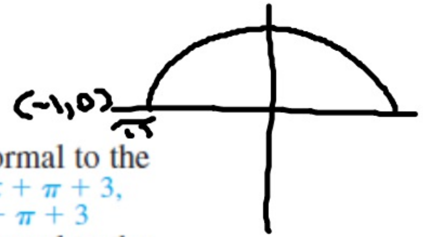
$s' = -2\sin t$

$s'' = -2\cos t$

$s''' = 2\sin t$

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\csc x = -\csc x\cot x$$
$$\frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\sec x = \sec x\tan x$$
$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

(skip finding the normal part)



21. Find equations for the lines that are tangent and normal to the graph of $y = \sin x + 3$ at $x = \pi$. tangent: $y = -x + \pi + 3$, normal: $y = x - \pi + 3$

22. Find equations for the lines that are tangent and normal to the graph of $y = \sec x$ at $x = \pi/4$. tangent: $y = 1.414x + 0.303$, normal: $y = -0.707x + 1.970$

23. Find equations for the lines that are tangent and normal to the graph of $y = x^2 \sin x$ at $x = 3$. tangent: $y = -8.063x + 25.460$, normal: $y = 0.124x + 0.898$

(21) $y = \sin x + 3$ at $x = \pi$

$y' = \cos x$ $\qquad y'(\pi) = \cos(\pi) = -1$

$m_{tan} = -1$

$y = 0 + 3$

$y_1 = 3$

$y - y_1 = m(x - x_1)$

$y - 3 = -1(x - \pi)$

$y - 3 = -x + \pi$
$+3 \qquad\qquad +3$

$y = -x + \pi + 3$

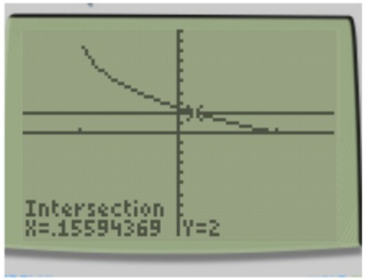
HW: p. 146 #1-30

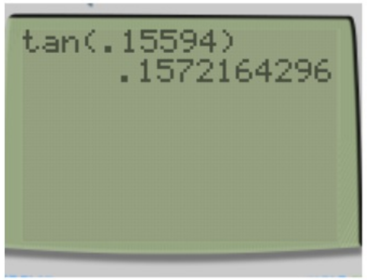
$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

$$\frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

30. Find the points on the curve $y = \tan x$, $-\pi/2 < x < \pi/2$, where the tangent is parallel to the line $y = 2x$. See page 147.





perhaps graphing will help...

$y = \tan x$

$y' = \underbrace{\sec^2 x}_{Y_1} = \underbrace{2}_{Y_2}$

$x = .15594$

$y = .1572$