3.6 The Chain Rule "decompose"

EXAMPLE 1 Relating Derivatives

The function $y = 6x - 10 = 2(3x - 5)$ is the composite of the functions $y = 2u$ and $u = 3x - 5$. How are the derivatives of these three functions related?

$\frac{d}{dx}[y = 6x - 10]$ $\frac{d}{du}[y = 2u]$ $\frac{du}{dx}[u = 3x - 5]$

SOLUTION

We have

$$\frac{dy}{dx} = 6, \quad \frac{dy}{du} = 2, \quad \frac{du}{dx} = 3.$$

Since $6 = 2 \cdot 3$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Now try Exercise 1.

EXAMPLE 2 Relating Derivatives

The polynomial $y = 9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$ is the composite of $y = u^2$ and $u = 3x^2 + 1$. Calculating derivatives, we see that

$$\begin{aligned}
\frac{dy}{du} \cdot \frac{du}{dx} &= 2u \cdot 6x \\
&= 2(3x^2 + 1) \cdot 6x \\
&= 36x^3 + 12x.
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(9x^4 + 6x^2 + 1) \\
&= 36x^3 + 12x.
\end{aligned}$$

Once again,

$$\frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{dx}.$$

Now try Exercise 5.

37. $f(u) = \dfrac{2u}{u^2+1}, \quad u = g(x) = 10x^2 + x + 1, \quad x = 0$ 0

$u = 1$

$$\frac{(u^2+1)(2)-(2u)(2u)}{(u^2+1)^2}$$

$$= \frac{2u^2+2-4u^2}{(u^2+1)^2} = \frac{-2u^2+2}{(u^2+1)^2} = \frac{0}{4} = 0$$

$$g'x = 20x+1$$
$$= 20(0)+1$$
$$= 1$$

$$g(u) = 10x^2 + x + 1$$
$$= 1$$

In Exercises 1–8, use the given substitution and the Chain Rule to find dy/dx.

1. $y = \sin(3x + 1)$, $u = 3x + 1$ — $3\cos(3x + 1)$
2. $y = \sin(7 - 5x)$, $u = 7 - 5x$ — $-5\cos(7 - 5x)$
3. $y = \cos(\sqrt{3}x)$, $u = \sqrt{3}x$
4. $y = \tan(2x - x^3)$, $u = 2x - x^3$
5. $y = \left(\dfrac{\sin x}{1 + \cos x}\right)^2$, $u = \dfrac{\sin x}{1 + \cos x}$ — $\dfrac{2\sin x}{(1 + \cos x)^2}$
6. $y = 5\cot\left(\dfrac{2}{x}\right)$, $u = \dfrac{2}{x}$
7. $y = \cos(\sin x)$, $u = \sin x$ — $-\sin(\sin x)\cos x$
8. $y = \sec(\tan x)$, $u = \tan x$ — $\sec(\tan x)\tan(\tan x)\sec^2 x$

$3\cos(u)$

1 $y = \sin(u)$ $u = 3x + 1$

$\dfrac{dy}{du} = \cos(u)$ $\dfrac{du}{dx} = 3$

$\dfrac{dy}{du} \cdot \dfrac{du}{dx}$

In Exercises 1–8, use the given substitution and the Chain Rule to find dy/dx.

$3\cos(3x+1)$

1. $y = \sin(3x+1),\ u = 3x+1$
2. $y = \sin(7-5x),\ u = 7-5x$
3. $y = \cos(\sqrt{3}x),\ u = \sqrt{3}x$
4. $y = \tan(2x - x^3),\ u = 2x - x^3$
5. $y = \left(\frac{\sin x}{1+\cos x}\right)^2,\quad u = \frac{\sin x}{1+\cos x}$ $\quad \frac{2\sin x}{(1+\cos x)^2}$
6. $y = 5\cot\left(\frac{2}{x}\right),\ u = \frac{2}{x}$
7. $y = \cos(\sin x),\ u = \sin x$
8. $y = \sec(\tan x),\ u = \tan x$

$y = \sin(u) \qquad u = 7-5x$

2

$\frac{dy}{du} = \cos(u) \qquad \frac{du}{dx} = -5$

$-5\cos(7-5x)$

5. $y = \left(\dfrac{\sin x}{1 + \cos x}\right)^2$, $u = \dfrac{\sin x}{1 + \cos x}$ $\dfrac{2 \sin x}{(1 + \cos x)^2}$

$y = u^2$ $u = \dfrac{\sin x}{1 + \cos x}$

$\dfrac{dy}{du} = 2u$ $\dfrac{du}{dx} = \dfrac{(1+\cos x)(\cos x) - \sin x \cancel{\cos x}}{(1+\cos x)^2}$

$= \dfrac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$

$\dfrac{du}{dx} = \dfrac{\cos x + 1}{(1+\cos x)^2}$

$2\left(\dfrac{\sin}{1+\cancel{\cos x}}\right)\left(\dfrac{\cancel{\cos x}+1}{(1+\cos x)^2}\right) = \dfrac{2 \sin x}{(1+\cos x)^2}$

In Exercises 1–8, use the given substitution and the Chain Rule to find dy/dx.

1. $y = \sin(3x + 1)$, $u = 3x + 1$ 2. $y = \sin(7 - 5x)$, $u = 7 - 5x$
3. $y = \cos(\sqrt{3}x)$, $u = \sqrt{3}x$ 4. $y = \tan(2x - x^3)$, $u = 2x - x^3$
5. $y = \left(\dfrac{\sin x}{1 + \cos x}\right)^2$, $u = \dfrac{\sin x}{1 + \cos x}$
6. $y = 5\cot\left(\dfrac{2}{x}\right)$, $u = \dfrac{2}{x}$ 7. $y = \cos(\sin x)$, $u = \sin x$
8. $y = \sec(\tan x)$, $u = \tan x$

$\frac{dy}{dx}$? $\frac{dy}{du} \cdot \frac{du}{dx}$

① $y = \sin(3x+1)$

$y = \sin(u)$ $u = 3x+1$

$\frac{dy}{du} = \cos(u)$ $\frac{du}{dx} = 3$ $f(g(x))$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos(3x+1) \cdot 3$

$= 3\cos(3x+1)$

5. $y = \left(\dfrac{\sin x}{1 + \cos x}\right)^2$, $u = \dfrac{\sin x}{1 + \cos x}$

$y = \left(\frac{\sin x}{1+\cos x}\right)^2$

$y = u^2$ $u = \frac{\sin x}{1+\cos x}$ f g

$\frac{dy}{du} = 2u$

$\frac{dy}{dx} = 2\left(\frac{\sin x}{1+\cos x}\right)\left(\frac{1}{1+\cos x}\right)$

$= \frac{2\sin x}{(1+\cos x)^2}$

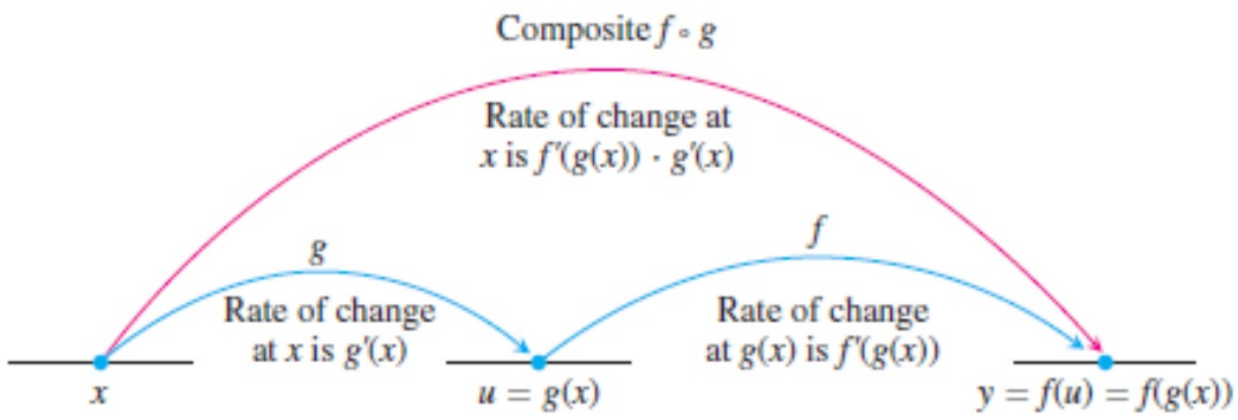
$f = \sin x$ $g = 1+\cos x$

$f' = \cos x$ $g' = -\sin x$

$\frac{du}{dx} = \frac{(1+\cos x)(\cos x) - \sin x(-\sin x)}{(1+\cos x)^2}$

$= \frac{\cos x + \underbrace{\cos^2 x + \sin^2 x}_{=1}}{(1+\cos x)^2} = \frac{\cancel{1+\cos x}}{(1+\cos x)^{\cancel{2}}}$

$\frac{du}{dx} = \frac{1}{1+\cos x}$



RULE 8 The Chain Rule

If f is differentiable at the point $u = g(x)$, and g is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

The trick; we need to see the functions as a composite, *f(g(x)),* and we'll need to "decompose" it.

EXAMPLE 3 Applying the Chain Rule

An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t.

SOLUTION

We know that the velocity is dx/dt. In this instance, x is a composite function: $x = \cos(u)$ and $u = t^2 + 1$. We have

$$\frac{dx}{du} = -\sin(u) \qquad x = \cos(u)$$

$$\frac{du}{dt} = 2t. \qquad u = t^2 + 1$$

By the Chain Rule,

$$\begin{aligned} \frac{dx}{dt} &= \frac{dx}{du} \cdot \frac{du}{dt} \\ &= -\sin(u) \cdot 2t \\ &= -\sin(t^2 + 1) \cdot 2t \\ &= -2t \sin(t^2 + 1). \end{aligned}$$

Now try Exercise 9.

In Exercises 9–12, an object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = s(t)$. Find the velocity of the object as a function of t.

$\frac{d}{dx}(\pi - 4t) = -4$

9. $s = \cos\left(\frac{\pi}{2} - 3t\right)$

10. $s = t\cos(\pi - 4t)$

11. $s = \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\cos 5t$

12. $s = \sin\left(\frac{3\pi}{2}t\right) + \cos\left(\frac{7\pi}{4}t\right)$

9 $\quad s = \cos(u) \quad u = \frac{\pi}{2} - 3t$

$\frac{ds}{du} = -\sin(u) \qquad \frac{du}{dt} = -3$

$v(t) = \frac{ds}{dt} = 3\sin\left(\frac{\pi}{2} - 3t\right)$

In Exercises 9–12, an object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = s(t)$. Find the velocity of the object as a function of t.

9. $s = \cos\left(\frac{\pi}{2} - 3t\right)$

10. $s = t\cos(\pi - 4t)$ $\quad 4t\sin(\pi - 4t) + \cos(\pi - 4t)$

11. $s = \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\cos 5t$ $\quad \frac{4}{\pi}\cos 3t - \frac{4}{\pi}\sin 5t$

12. $s = \sin\left(\frac{3\pi}{2}t\right) + \cos\left(\frac{7\pi}{4}t\right)$ $\quad \frac{3\pi}{2}\cos\frac{3\pi t}{2} - \frac{7\pi}{4}\sin\frac{7\pi t}{4}$

10 $\quad s = t\cos(\pi - 4t)$

$f \quad g$

$f = t$

$f' = 1$

$g = \cos(\pi - 4t)$

$g' = -4(-\sin(\pi - 4t))$

$4\sin(\pi - 4t)$

$y = \cos(u) \quad u = \pi - 4t$

$y' = -\sin(u) \quad u' = -4$

“Outside-Inside” Rule

It sometimes helps to think about the Chain Rule this way: If $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

In words, differentiate the “outside” function f and evaluate it at the “inside” function $g(x)$ left alone; then multiply by the derivative of the “inside function.”

EXAMPLE 4 Differentiating from the Outside in

Differentiate $\sin(x^2 + x)$ with respect to x.

SOLUTION

$$\frac{d}{dx}\sin\underbrace{(x^2 + x)}_{\text{inside}} = \cos\underbrace{(x^2 + x)}_{\substack{\text{inside} \\ \text{left alone}}} \cdot \underbrace{(2x + 1)}_{\substack{\text{derivative of} \\ \text{the inside}}}$$

Now try Exercise 13.

In Exercises 13–24, find dy/dx. If you are unsure of your answer, use NDER to support your computation.

13. $y = (x + \sqrt{x})^{-2}$

14. $y = (\csc x + \cot x)^{-1}$

15. $y = \sin^{-5} x - \cos^3 x$

16. $y = x^3(2x - 5)^4$

17. $y = \sin^3 x \tan 4x$

18. $y = 4\sqrt{\sec x + \tan x}$

19. $y = \dfrac{3}{\sqrt{2x + 1}}$

20. $y = \dfrac{x}{\sqrt{1 + x^2}}$

21. $y = \sin^2 (3x - 2)$

22. $y = (1 + \cos 2x)^2$

23. $y = (1 + \cos^2 7x)^3$

24. $y = \sqrt{\tan 5x}$

$(\sin x)^{-5}$ $(\cos x)^3$

In Exercises 13–24, find dy/dx. If you are unsure of your answer, use NDER to support your computation.

13. $y = (x + \sqrt{x})^{-2}$
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19. $y = \dfrac{3}{\sqrt{2x + 1}}$
20. $y = \dfrac{x}{\sqrt{1 + x^2}}$
21. $y = \sin^2 (3x - 2)$
22. $y = (1 + \cos 2x)^2$
23. $y = (1 + \cos^2 7x)^3$
24. $y = \sqrt{\tan 5x}$

$\frac{d}{dx} \sec x = \sec x \tan x$

$\frac{d}{dx} \tan x = \sec^2 x$

18 $\quad y = 4\sqrt{\sec x + \tan x}$

$u = \sec x + \tan x$

$y = 4\sqrt{u}$
$= 4u^{1/2}$

$\frac{dy}{du} = 2u^{-1/2}$

$\frac{du}{dx} = \sec x \tan x + \sec^2 x$

$\frac{dy}{du} \cdot \frac{du}{dx}$

$\frac{dy}{dx} = 2(\sec x + \tan x)^{-1/2} \cdot (\sec x \tan x + \sec^2 x)$

In Exercises 25–28 find $dr/d\theta$.

25. $r = \tan(2 - \theta)$

26. $r = \sec 2\theta \tan 2\theta$

27. $r = \sqrt{\theta \sin \theta}$

28. $r = 2\theta\sqrt{\sec \theta}$

In Exercises 29–32, find y''.

29. $y = \tan x$

30. $y = \cot x$

31. $y = \cot(3x - 1)$

32. $y = 9 \tan(x/3)$

In Exercises 33–38, find the value of $(f \circ g)'$ at the given value of x.

33. $f(u) = u^5 + 1, \quad u = g(x) = \sqrt{x}, \quad x = 1$ 5/2

34. $f(u) = 1 - \dfrac{1}{u}, \quad u = g(x) = \dfrac{1}{1 - x}, \quad x = -1$ 1

35. $f(u) = \cot \dfrac{\pi u}{10}, \quad u = g(x) = 5\sqrt{x}, \quad x = 1$ $-\pi/4$

36. $f(u) = u + \dfrac{1}{\cos^2 u}, \quad u = g(x) = \pi x, \quad x = \dfrac{1}{4}$ 5π

37. $f(u) = \dfrac{2u}{u^2 + 1}, \quad u = g(x) = 10x^2 + x + 1, \quad x = 0$ 0

38. $f(u) = \left(\dfrac{u - 1}{u + 1}\right)^2, \quad u = g(x) = \dfrac{1}{x^2} - 1, \quad x = -1$ -8

What happens if you can write a function as a composite in different ways? Do you get the same derivative each time? The Chain Rule says you should. Try it with the functions in Exercises 39 and 40.

39. Find dy/dx if $y = \cos(6x + 2)$ by writing y as a composite with

(a) $y = \cos u$ and $u = 6x + 2$.

(b) $y = \cos 2u$ and $u = 3x + 1$

40. Find dy/dx if $y = \sin(x^2 + 1)$ by writing y as a composite with

(a) $y = \sin(u + 1)$ and $u = x^2$.

(b) $y = \sin u$ and $u = x^2 + 1$.