3.7 Implicit Differentiation

Explicit verses implicit; what's the difference?

Ex. A function where y is expressed *explicitly;*

$$\frac{d}{dx}\left[y = 3x + 4\right]$$

$$\frac{dy}{dx} = 3$$

Ex. A function where y is expressed *implicitly;*

$$x^3y + 4 = 6x$$

in this context, "implicit" means "hidden."

BTW, we don't always have to derive implicitly...

EXAMPLE 1 Differentiating Implicitly

Find dy/dx if $y^2 = x$.

SOLUTION

To find dy/dx, we simply differentiate both sides of the equation $y^2 = x$ with respect to x, treating y as a differentiable function of x and applying the Chain Rule:

$$y^2 = x$$

$$2y\frac{dy}{dx} = 1 \qquad \frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\cdot\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2y}.$$

Now try Exercise 3.

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

we can apply the chain rule to help understand this...

$$\frac{d}{dx}[y^2] = \frac{d}{dx}[x]$$

variables match!

don't match

$u = y$

$f(x) = u^2$

$f' = 2u$

$$\frac{dy}{du} = 2y\cdot\frac{dy}{dx} = 1$$

$$\frac{d}{dx}[u] = \frac{d}{dx}[y]$$

$$\frac{du}{dx} = \frac{dy}{dx}$$

...sometimes, though, you *have* to derive implicitly.
(what if we can't solve for y?)

Implicit Differentiation Process

1. Differentiate both sides of the equation with respect to x.
2. Collect the terms with dy/dx on one side of the equation.
3. Factor out dy/dx.
4. Solve for dy/dx.

Think algebraic; to solve for dy/dx, get it by itself!

$f = x \quad g = y$

$f' = 1 \quad g' = \frac{dy}{dx}$

$$x^2 - xy + y^2 = 7$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}\overset{fg}{(xy)} + \frac{d}{dx}(y^2) = \frac{d}{dx}(7)$$ Differentiate both sides with respect to x . . .

$$2x - \underbrace{\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right)}_{fg' + gf'} + 2y\frac{dy}{dx} = 0$$ . . . treating xy as a product and y as a function of x.

$$(2y - x)\frac{dy}{dx} = y - 2x$$ Collect terms.

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}.$$ Solve for dy/dx.

4. $\frac{y}{x} - (x+y)^2$ or $\frac{1-3x^2-2xy}{x^2+1}$

In Exercises 1–8, find dy/dx.

1. $x^2y + xy^2 = 6$ $\quad -\frac{2xy+y^2}{2xy+x^2}$
2. $x^3 + y^3 = 18xy$ $\quad \frac{6y-x^2}{y^2-6x}$
3. $y^2 = \frac{x-1}{x+1}$ $\quad \frac{1}{y(x+1)^2}$
4. $x^2 = \frac{x-y}{x+y}$
5. $x = \tan y$ $\quad \cos^2 y$
6. $x = \sin y$ $\quad \sec y$
7. $x + \tan(xy) = 0$ See page 164.
8. $x + \sin y = xy$ $\quad \frac{1-y}{x-\cos y}$

1 $\frac{d}{dx}x^2y + \frac{d}{dx}xy^2 = \frac{d}{dx}6$ $\quad$ 1 differentiate

$f = x^2 \quad g = y$

$f' = 2x \quad g' = 1\cdot\frac{dy}{dx}$

$f = x \quad g = y^2$

$f' = 1 \quad g' = 2y\cdot\frac{dy}{dx}$

$$2xy + x^2\frac{dy}{dx} + y^2 + 2xy\cdot\frac{dy}{dx} = 0$$

$$x^2\cdot\frac{dy}{dx} + 2xy\cdot\frac{dy}{dx} = -2xy - y^2$$

$$\frac{\frac{dy}{dx}(x^2+2xy)}{x^2+2xy} = \frac{-2xy-y^2}{x^2+2xy}$$

$$\frac{dy}{dx} = \frac{-2xy-y^2}{x^2+2xy}$$

In Exercises 1–8, find dy/dx.

product rule...

1. $x^2y + xy^2 = 6$
2. $x^3 + y^3 = 18xy$
3. $y^2 = \dfrac{x-1}{x+1}$
4. $x^2 = \dfrac{x-y}{x+y}$
5. $x = \tan y$
6. $x = \sin y$
7. $x + \tan(xy) = 0$
8. $x + \sin y = xy$

$1 = \cos y \cdot \dfrac{dy}{dx}$

$\dfrac{1}{\cos y} = \dfrac{\cancel{\cos y}}{\cos y} \cdot$

$= \sec y$!

(3) $\sqrt{y^2} = \sqrt{\dfrac{x-1}{x+1}}$

$y = \pm\sqrt{\dfrac{x-1}{x+1}}$

I can now use rules from 3.6....

OR

$y^2 = \dfrac{x-1}{x+1}$

$\dfrac{d}{dx}[y^2] = \dfrac{d}{dx}\left[\dfrac{x-1}{x+1}\right]$ f g

$\dfrac{\cancel{2y}\cdot\frac{dy}{dx}}{\cancel{2y}} = \dfrac{\frac{2}{(x+1)^2}}{2y}$

$x+1-x+1$

$= \dfrac{(x+1)(1)-(x-1)(1)}{(x+1)^2}$

$= \dfrac{2}{(x+1)^2}$

$\dfrac{dy}{dx} = \dfrac{2}{2y(x+1)^2} = \dfrac{1}{y(x+1)^2}$

7. $x + \tan(xy) = 0$

$$\frac{d}{dx}\tan(u) = \sec^2(u)\cdot u'$$

$$u = xy$$

$$u' = (1)(y) + \frac{dy}{dx}(x)$$

$$f = x \qquad g = y$$

$$f' = 1 \qquad g' = \frac{dy}{dx}$$

$$1 + \sec^2(xy)\cdot\left(y + \frac{dy}{dx}\cdot x\right) = 0$$

$$-1 \qquad\qquad -1$$

$$\cos^2(xy)\cdot \cancel{\sec^2}(xy)\left(y + \frac{dy}{dx}\cdot x\right) = -1\cdot\cos^2 xy$$

$$\frac{\cancel{3}}{\cancel{4}}\cdot\frac{\cancel{4}}{\cancel{3}}$$

$$y + \frac{dy}{dx}\cdot x = -\cos^2 xy$$

$$-y \qquad\qquad -y$$

$$\frac{dy}{dx}\cdot x = -\cos^2(xy) - y$$

$$\frac{dy}{dx} = \frac{-\cos^2(xy) - y}{x}$$

EXAMPLE 2 Finding Slope on a Circle

Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

SOLUTION

The circle is not the graph of a single function of x, but it is the union of the graphs of two differentiable functions, $y_1 = \sqrt{25 - x^2}$ and $y_2 = -\sqrt{25 - x^2}$ (Figure 3.49). The point $(3, -4)$ lies on the graph of y_2, so it is possible to find the slope by calculating explicitly:

$$\left.\frac{dy_2}{dx}\right|_{x=3} = -\left.\frac{-2x}{2\sqrt{25 - x^2}}\right|_{x=3} = -\frac{-6}{2\sqrt{25 - 9}} = \frac{3}{4}.$$

But we can also find this slope more easily by differentiating both sides of the equation of the circle implicitly with respect to x:

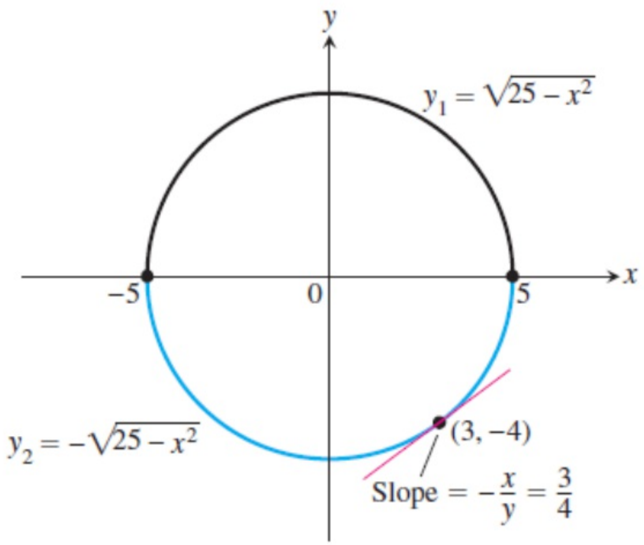
$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25) \quad \text{Differentiate both sides with respect to } x.$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

The slope at $(3, -4)$ is

$$-\left.\frac{x}{y}\right|_{(3,\,-4)} = -\frac{3}{-4} = \frac{3}{4}$$



Note: when evaluating, plug in values for x AND y.

In Exercises 9–12, find dy/dx and find the slope of the curve at the indicated point.

$\frac{dy}{dx} = -\frac{x}{y}$, 2/3

9. $x^2 + y^2 = 13, \quad (-2, 3)$

10. $x^2 + y^2 = 9, \quad (0, 3)$ $\quad \frac{dy}{dx} = -\frac{x}{y}, 0$

11. $(x-1)^2 + (y-1)^2 = 13, \quad (3, 4)$ $\quad \frac{dy}{dx} = -\frac{x-1}{y-1}, -2/3$

12. $(x+2)^2 + (y+3)^2 = 25, \quad (1, -7)$ See page 164.

9. $\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}13$

$2x + 2y\cdot\frac{dy}{dx} = 0$

$-2x \qquad -2x$

$\frac{2y\frac{dy}{dx}}{2y} = \frac{-2x}{2y}$

$\frac{dy}{dx} = \frac{-x}{y}$

$= -\frac{(-2)}{3}$

$= 2/3$

$$\frac{d}{dx}$$

In Exercises 9–12, find dy/dx and find the slope of the curve at the indicated point.

9. $x^2 + y^2 = 13, \quad (-2, 3)$

10. $x^2 + y^2 = 9, \quad (0, 3)$

11. $(x - 1)^2 + (y - 1)^2 = 13, \quad (3, 4)$

12. $(x + 2)^2 + (y + 3)^2 = 25, \quad (1, -7)$

12

$$\frac{2(x+2)(1)}{2} + \frac{2(y+3)\left(\frac{dy}{dx}\right)}{2} = \frac{0}{2}$$

$$(x+2) + (y+3)\left(\frac{dy}{dx}\right) = 0$$

$$\frac{\cancel{(y+3)}}{\cancel{(y+3)}}\left(\frac{dy}{dx}\right) = \frac{-(x+2)}{(y+3)} = \frac{-3}{-4} = \frac{3}{4}$$

EXAMPLE 3 Solving for *dy/dx*

Show that the slope dy/dx is defined at every point on the graph of $2y = x^2 + \sin y$.

SOLUTION

First we need to know dy/dx, which we find by implicit differentiation:

$$2y = x^2 + \sin y$$

$$\frac{d}{dx}(2y) = \frac{d}{dx}(x^2 + \sin y)$$ Differentiate both sides with respect to *x* ...

$$= \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin y)$$

$$2\frac{dy}{dx} = 2x + \cos y \frac{dy}{dx}$$ ... treating y as a function of *x* and using the Chain Rule.

$$2\frac{dy}{dx} - (\cos y)\frac{dy}{dx} = 2x$$ Collect terms with *dy/dx* ...

$$(2 - \cos y)\frac{dy}{dx} = 2x$$ and factor out *dy/dx*.

$$\frac{dy}{dx} = \frac{2x}{2 - \cos y}.$$ Solve for *dy/dx* by dividing.

The formula for dy/dx is defined at every point (x, y), except for those points at which $\cos y = 2$. Since $\cos y$ cannot be greater than 1, this never happens.

Now try Exercise 13.

In Exercises 13–16, find where the slope of the curve is defined.

13. $x^2y - xy^2 = 4$

14. $x = \cos y$

15. $x^3 + y^3 = xy$

16. $x^2 + 4xy + 4y^2 - 3x = 6$

it'll be easier to find where the derivative is UNdefined...

In Exercises 17–26, find the lines that are (**a**) tangent and (**b**) normal to the curve at the given point.

17. $x^2 + xy - y^2 = 1, \quad (2, 3)$ (a) $y = \frac{7}{4}x - \frac{1}{2}$ (b) $y = -\frac{4}{7}x + \frac{29}{7}$

18. $x^2 + y^2 = 25, \quad (3, -4)$ (a) $y = \frac{3}{4}x - \frac{25}{4}$ (b) $y = -\frac{4}{3}x$

19. $x^2y^2 = 9, \quad (-1, 3)$

See page 164.

In Exercises 13–16, find where the slope of the curve is defined.

$\frac{d}{dx}$ **13.** $x^2y - xy^2 = 4$ $\frac{d}{dx}$

14. $x = \cos y$

15. $x^3 + y^3 = xy$

16. $x^2 + 4xy + 4y^2 - 3x = 6$

$\frac{dy}{dx} = y'$

it'll be easier to find where the derivative is UNdefined...

13

$f = x^2 \quad g = y \quad g = u \quad u = y$

$f' = 2x \quad g' = 1 \cdot \frac{dy}{dx}$

$f = x \quad g = y^2$

$f' = 1 \cdot \quad g' = 2y \cdot \frac{dy}{dx}$

$Ax + By = C$

$$2x \cdot y + \frac{dy}{dx} \cdot x^2 - \left(1 \cdot y^2 + 2y\frac{dy}{dx} \cdot x\right) = 0$$

$$\cancel{2xy} + x^2y' - \cancel{y^2} - 2xyy' = 0$$

$$-2xy \qquad +y^2 \qquad -2xy + y^2$$

$$x^2y' - 2xyy' = -2xy + y^2$$

$$\frac{y'(x^2 - 2xy)}{x^2 - 2xy} = \frac{-2xy + y^2}{x^2 - 2xy}$$

$(-\infty, 0) \cup (0, \infty)$

$$y' = \frac{-2xy + y^2}{x(x - 2y)}$$

$x \neq 0$

$x - 2y \neq 0$

$x \neq 2y$

In Exercises 17–26, find the lines that are (a) tangent and (b) normal to the curve at the given point.

17. $x^2 + xy - y^2 = 1, \quad (2, 3)$ (a) $y = \frac{7}{4}x - \frac{1}{2}$ (b) $y = -\frac{4}{7}x + \frac{29}{7}$ do part (a) only...

18. $x^2 + y^2 = 25, \quad (3, -4)$ (a) $y = \frac{3}{4}x - \frac{25}{4}$ (b) $y = -\frac{4}{3}x$

19. $x^2y^2 = 9, \quad (-1, 3)$

$x \quad y$

$\frac{dy}{dx} = y'$

19a $x^2 y^2 = 9$

$f \quad g$

$f = x^2 \quad g = y^2$

$f' = 2x \quad g' = 2y\frac{dy}{dx}$

$(2x)(y^2) + \left(2y\frac{dy}{dx}\right)(x^2) = 0$

$2xy^2 + 2x^2y\,y' = 0$
$-2xy^2 \qquad -2xy^2$

$\frac{2x^2y\,y'}{2x^2y} = \frac{-2xy^2}{2x^2y}$

$\frac{dy}{dx} = -\frac{y}{x}$

$m = \frac{-(3)}{(-1)} = 3$

$y - 3 = 3(x - (-1))$

$y - 3 = 3(x + 1)$

$y - 3 = 3x + 3$
$+3 \qquad +3$

$\boxed{y = 3x + 6}$

EXAMPLE 5 Finding a Second Derivative Implicitly

Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

SOLUTION

To start, we differentiate both sides of the equation with respect to x in order to find $y' = dy/dx$.

$$\frac{d}{dx}(2x^3 - 3y^2) = \frac{d}{dx}(8)$$

$$6x^2 - 6yy' = 0$$

$$x^2 - yy' = 0$$

$$y' = \frac{x^2}{y}, \text{ when } y \neq 0$$

We now apply the Quotient Rule to find y''.

$$y'' = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2} \cdot y'$$

Finally, we substitute $y' = x^2/y$ to express y'' in terms of x and y.

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right) = \frac{2x}{y} - \frac{x^4}{y^3}, \text{ when } y \neq 0$$

Now try Exercise 29.

In Exercises 27–30, use implicit differentiation to find dy/dx and then d^2y/dx^2.

27. $x^2 + y^2 = 1$ See page 164.

28. $x^{2/3} + y^{2/3} = 1$ See page 164.

29. $y^2 = x^2 + 2x$ See page 164.

30. $y^2 + 2y = 2x + 1$ See page 164.

In Exercises 27–30, use implicit differentiation to find dy/dx and then d^2y/dx^2.

27. $x^2 + y^2 = 1$ **28.** $x^{2/3} + y^{2/3} = 1$

29. $y^2 = x^2 + 2x$ **30.** $y^2 + 2y = 2x + 1$

(28) $x^{2/3} + y^{2/3} = 1$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3} \cdot y' = 0$$

$$\frac{2}{3x^{1/3}} + \frac{2y'}{3y^{1/3}} = 0$$

$$\frac{3y^{1/3}}{2} \cdot \frac{2y'}{3y^{1/3}} = -\frac{2}{3x^{1/3}} \cdot \frac{3y^{1/3}}{2}$$

w.r.t. x....

$f = -y^{1/3}$

$f' = -\frac{1}{3}y^{-2/3} \cdot y'$

$$y' = \frac{-y^{1/3}}{x^{1/3}} \begin{matrix} \leftarrow f \\ \leftarrow g \end{matrix}$$

$$y'' = \frac{x^{1/3}\left(-\frac{1}{3}y^{-2/3}\right)y' - \left(-y^{1/3}\right)\left(\frac{1}{3}x^{-2/3}\right)}{\left(x^{1/3}\right)^2}$$

$$y'' = \frac{\cancel{x^{1/3}}\left(+\frac{1}{3}y^{-2/3}\right)\left(\frac{+y^{1/3}}{\cancel{x^{1/3}}}\right) - \left(-y^{1/3}\right)\left(\frac{1}{3}x^{-2/3}\right)}{\left(x^{1/3}\right)^2}$$

$$y'' = \frac{\dfrac{1y^{1/3}}{3y^{2/3}} + \dfrac{y^{1/3}}{3x^{2/3}}}{x^{2/3}}$$

:)

In Exercises 31–42, find dy/dx.

31. $y = x^{9/4}$ $(9/4)x^{5/4}$

32. $y = x^{-3/5}$ $(-3/5)x^{-8/5}$

33. $y = \sqrt[3]{x}$ $(1/3)x^{-2/3}$

34. $y = \sqrt[4]{x}$ $(1/4)x^{-3/4}$

35. $y = (2x + 5)^{-1/2}$ $-(2x + 5)^{-3/2}$

36. $y = (1 - 6x)^{2/3}$ $-4(1 - 6x)^{-1/3}$

37. $y = x\sqrt{x^2 + 1}$
$x^2(x^2 + 1)^{-1/2} + (x^2 + 1)^{1/2}$

38. $y = \dfrac{x}{\sqrt{x^2 + 1}}$ $(x^2 + 1)^{-3/2}$

39. $y = \sqrt{1 - \sqrt{x}}$

40. $y = 3(2x^{-1/2} + 1)^{-1/3}$

41. $y = 3(\csc x)^{3/2}$

42. $y = [\sin (x + 5)]^{5/4}$