3.8 Inverse Trigonometric Functions

$f(x) = \arcsin(x)$

$f(x) = \sin^{-1}(x)$

Let's derive y=sin^{-1}(x)

$\frac{d}{dx}[x] = \frac{d}{dx}[\sin y]$ $\qquad x = \sin y$

$1 = \cos y \cdot \frac{dy}{dx}$

$\frac{1}{\cos y} \qquad \frac{\cancel{\cos y}}{\cos y}$

(The Pythagorean Theorem
will be helpful here)

$\frac{dy}{dx} = \pm\frac{1}{\sqrt{1-\sin^2 y}}$ $\qquad \sin^2 y + \cos^2 y = 1$

$\frac{d}{dx}[\sin^{-1}x] = \pm\frac{1}{\sqrt{1-x^2}}$ $\qquad \cos^2 y = 1 - \sin^2 y$

$\cos y = \pm\sqrt{1-\sin^2 y}$

$$\boxed{\frac{d}{dx}[\arcsin x] = \pm\frac{1}{\sqrt{1-x^2}}}$$

Now, you try to find the derivative of arccos...

$$y = \cos^{-1} x$$

$$\frac{d}{dx}(x) = \frac{d}{dx}(\cos y)$$

$$\sin^2 y + \cos^2 y = 1$$

$$\sin^2 y = 1 - \cos^2 y$$

$$\sin y = \pm\sqrt{1-\cos^2 y}$$

$$\frac{1}{-\sin y} = \frac{-\sin y \cdot \frac{dy}{dx}}{-\sin y}$$

$$\frac{dy}{dx} = -\frac{1}{\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}}$$

$$= \pm\frac{1}{\sqrt{1-x^2}}$$

so...

$$\boxed{\frac{d}{dx}\left[\arccos x\right] = \pm\frac{1}{\sqrt{1-x^2}}}$$

in a similar manner, the other trig functions can be found...

$$\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\csc^{-1}x\right) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\cos^{-1}x\right) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\sec^{-1}x\right) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left(\cot^{-1}x\right) = -\frac{1}{1+x^2}$$

Furthermore, we can generalize any type of trig function by using the Chain Rule;

$$\frac{dx}{dx} = \frac{dx}{du} \cdot \frac{du}{dx}$$

13.14 $\frac{d}{dx} \sin u = \cos u \frac{du}{dx}$

13.15 $\frac{d}{dx} \cos u = -\sin u \frac{du}{dx}$

13.16 $\frac{d}{dx} \tan u = \sec^2 u \frac{du}{dx}$

13.17 $\frac{d}{dx} \cot u = -\csc^2 u \frac{du}{dx}$

13.18 $\frac{d}{dx} \sec u = \sec u \tan u \frac{du}{dx}$

13.19 $\frac{d}{dx} \csc u = -\csc u \cot u \frac{du}{dx}$

13.20 $\frac{d}{dx} \sin^{-1} u = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx} \qquad \left[-\frac{\pi}{2} < \sin^{-1} u < \frac{\pi}{2}\right]$

13.21 $\frac{d}{dx} \cos^{-1} u = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx} \qquad [0 < \cos^{-1} u < \pi]$

13.22 $\frac{d}{dx} \tan^{-1} u = \frac{1}{1+u^2} \frac{du}{dx} \qquad \left[-\frac{\pi}{2} < \tan^{-1} u < \frac{\pi}{2}\right]$

13.23 $\frac{d}{dx} \cot^{-1} u = \frac{-1}{1+u^2} \frac{du}{dx} \qquad [0 < \cot^{-1} u < \pi]$

13.24 $\frac{d}{dx} \sec^{-1} u = \frac{1}{|u|\sqrt{u^2-1}} \frac{du}{dx} = \frac{\pm 1}{u\sqrt{u^2-1}} \frac{du}{dx} \qquad \left[\begin{array}{l} + \text{ if } 0 < \sec^{-1} u < \pi/2 \\ - \text{ if } \pi/2 < \sec^{-1} u < \pi \end{array}\right]$

13.25 $\frac{d}{dx} \csc^{-1} u = \frac{-1}{|u|\sqrt{u^2-1}} \frac{du}{dx} = \frac{\mp 1}{u\sqrt{u^2-1}} \frac{du}{dx} \qquad \left[\begin{array}{l} - \text{ if } 0 < \csc^{-1} u < \pi/2 \\ + \text{ if } -\pi/2 < \csc^{-1} u < 0 \end{array}\right]$

29. (a) f'

In Exercises 1–8, find the derivative of y with respect to the appropriate variable. 2

1. $y = \cos^{-1}(x^2)$ $\quad -\dfrac{2x}{\sqrt{1-x^4}}$

2. $y = \cos^{-1}(1/x)$ $\quad \dfrac{1}{|x|\sqrt{x^2-1}}$

3. $y = \sin^{-1}\sqrt{2}\,t$ $\quad \dfrac{\sqrt{2}}{\sqrt{1-2t^2}}$

4. $y = \sin^{-1}(1-t)$ $\quad -\dfrac{1}{\sqrt{2t-t^2}}$ 2

5. $y = \sin^{-1}\dfrac{3}{t^2}$ $\quad -\dfrac{6}{t\sqrt{t^4-9}}$

6. $y = s\sqrt{1-s^2} + \cos^{-1} s$ $\quad -\dfrac{2s^2}{\sqrt{1-s^2}}$

$\dfrac{d}{ds}!$

$f = s \qquad g = (1-s^2)^{\frac{1}{2}}$

$f' = 1 \qquad g' = \frac{1}{2}(1-s^2)^{-\frac{1}{2}} \cdot -2s$

$u^{\frac{1}{2}} \cdot u^{\frac{1}{2}} = u^1$

$\sqrt{u} \cdot \sqrt{u} = u$

$(1)(1-s^2)^{\frac{1}{2}} + \frac{1}{2}(1-s^2)^{-\frac{1}{2}} \cdot -2s \cdot s$

$(1-s^2)^{\frac{1}{2}} - \dfrac{s^2}{(1-s^2)^{\frac{1}{2}}} - \dfrac{1}{\sqrt{1-s^2}}$

$\dfrac{\sqrt{1-s^2} \cdot \sqrt{(1-s^2)}}{\sqrt{1-s^2}} - \dfrac{s^2}{\sqrt{1-s^2}} - \dfrac{1}{\sqrt{1-s^2}}$

$\dfrac{(1-s^2)}{\sqrt{1-s^2}} - \dfrac{s^2}{\sqrt{1-s^2}} - \dfrac{1}{\sqrt{1-s^2}}$

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

In Exercises 9–12, a particle moves along the x-axis so that its position at any time $t \ge 0$ is given by $x(t)$. Find the velocity at the indicated value of t.

9. $x(t) = \sin^{-1}\left(\frac{t}{4}\right)$, $t = 3$ $\quad \sqrt{7}/7$

10. $x(t) = \sin^{-1}\left(\frac{\sqrt{t}}{4}\right)$, $t = 4$ $\quad \sqrt{3}/24$

11. $x(t) = \tan^{-1} t$, $t = 2$ $\quad 1/5$

12. $x(t) = \tan^{-1}(t^2)$, $t = 1$ $\quad 1$

(10) $x(t) = \sin^{-1}\left(\frac{\sqrt{t}}{4}\right)$

$\frac{1}{4} t^{1/2}$

$u = \frac{\sqrt{t}}{4}$

$\frac{du}{dt} = \frac{1}{8} t^{-\frac{1}{2}} = \frac{1}{8\sqrt{t}}$

$x(u) = \sin^{-1}(u)$

$x'(u) = \frac{1}{\sqrt{1-u^2}} \cdot u'$

$= \frac{1}{\sqrt{1-\left(\frac{\sqrt{t}}{4}\right)^2}} \cdot \left(\frac{1}{8\sqrt{t}}\right)$

$= \frac{1}{8\sqrt{t}\cdot\sqrt{1-\frac{t}{16}}}$ $\leftarrow t=4$

$1 - \frac{4}{16} = \frac{12}{16} = \frac{3}{4}$

$\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

$= \frac{1}{8(2)\left(\frac{\sqrt{3}}{2}\right)}$

$= \frac{1}{8\sqrt{3}}$

$u \quad u\,u \quad \cdot u'$

In Exercises 13–22, find the derivatives of y with respect to the appropriate variable.

$u = 5s \ldots$

13. $y = \sec^{-1}(2s + 1)$ See page 171.

14. $y = \sec^{-1} 5s \quad \dfrac{1}{|s|\sqrt{25s^2 - 1}}$ $5s$ $\cdot 5$

15. $y = \csc^{-1}(x^2 + 1),\ x > 0$ See page 171.

16. $y = \csc^{-1} x/2 \quad -\dfrac{2}{|x|\sqrt{x^2 - 4}}$

17. $y = \sec^{-1}\dfrac{1}{t},\quad 0 < t < 1$ See page 171.

18. $y = \cot^{-1}\sqrt{t} \quad -\dfrac{1}{2\sqrt{t}(t + 1)}$

19. $y = \cot^{-1}\sqrt{t - 1}$ See page 171.

20. $y = \sqrt{s^2 - 1} - \sec^{-1} s \quad \dfrac{s|s| - 1}{|s|\sqrt{s^2 - 1}}$

21. $y = \tan^{-1}\sqrt{x^2 - 1} + \csc^{-1} x,\quad x > 1 \quad 0,\ x > 1$

22. $y = \cot^{-1}\dfrac{1}{x} - \tan^{-1} x \quad 0,\ x \neq 0$

(13) $y = \sec^{-1}(2s+1)$

$y = \sec^{-1} u \qquad u = 2s+1$

$y' = \dfrac{1}{u\sqrt{u^2-1}} \qquad u' = 2$

multiply...

$$\frac{dy}{ds} = \frac{1}{(2s+1)\sqrt{(2s+1)^2-1}} \cdot 2$$

reduce

(15) $y = \csc^{-1}(x^2+1)$

$u = x^2+1$
$u' = 2x$

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

$u = x^2+1$
$u' = 2x$

$$\frac{d}{dx}[\csc^{-1} u] = \frac{-1}{u\sqrt{u^2-1}} \cdot u'$$

$$= \frac{-1}{(x^2+1)\sqrt{(x^2+1)^2-1}} \cdot 2x$$

$$\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\csc^{-1}x\right) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\cos^{-1}x\right) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\sec^{-1}x\right) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left(\cot^{-1}x\right) = -\frac{1}{1+x^2}$$

skip!

In Exercises 23–26, find an equation for the tangent to the graph of y at the indicated point.

$y = 0.289x + 0.470$

23. $y = \sec^{-1}x, \quad x = 2$

$y = \frac{1}{5}x + 0.707$

24. $y = \tan^{-1}x, \quad x = 2$

25. $y = \sin^{-1}\left(\frac{x}{4}\right), \quad x = 3$

$y = 0.378x - 0.286$

26. $y = \tan^{-1}(x^2), \quad x = 1$

$y = x - 0.215$

In Exercises 1–8, find the derivative of y with respect to the appropriate variable.

1. $y = \cos^{-1}(x^2)$ $\quad -\dfrac{2x}{\sqrt{1-x^4}}$

2. $y = \cos^{-1}(1/x)$ $\quad \dfrac{1}{|x|\sqrt{x^2-1}}$

3. $y = \sin^{-1}\sqrt{2}\,t$ $\quad \dfrac{\sqrt{2}}{\sqrt{1-2t^2}}$

4. $y = \sin^{-1}(1-t)$ $\quad -\dfrac{1}{\sqrt{2t-t^2}}$

5. $y = \sin^{-1}\dfrac{3}{t^2}$ $\quad -\dfrac{6}{t\sqrt{t^4-9}}$

6. $y = s\sqrt{1-s^2} + \cos^{-1}s$ $\quad -\dfrac{2s^2}{\sqrt{1-s^2}}$

7. $y = x\sin^{-1}x + \sqrt{1-x^2}$ $\quad \sin^{-1}x$

8. $y = \dfrac{1}{\sin^{-1}(2x)}$ $\quad -\dfrac{2}{(\sin^{-1}2x)^2\sqrt{1-4x^2}}$

$u = x^2$

$\dfrac{du}{dx} = 2x$

$$\frac{d}{dx}\left[\cos^{-1}(u)\right] = \frac{-1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$$

(1) $y = \cos^{-1}(x^2)$

$$y' = \frac{-1}{\sqrt{1-(x^2)^2}}\cdot 2x$$

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

28. Let $f(x) = x^5 + 2x^3 + x - 1$.

(a) Find $f(1)$ and $f'(1)$. $f(1) = 3, f'(1) = 12$

(b) Find $f^{-1}(3)$ and $(f^{-1})'(3)$. $f^{-1}(3) = 1, (f^{-1})'(3) = \frac{1}{12}$

$f(x)$; $x = 1$, $y = 3$

b) 28

$y = x^5 + 2x^3 + x - 1$

$\frac{d}{dx}\left[x = y^5 + 2y^3 + y - 1\right]$

$f^{-1}(x)$; $x = 3$, $y = 1$

$1 = 5y^4\frac{dy}{dx} + 6y^2\frac{dy}{dx} + 1\cdot\frac{dy}{dx} - 0$

$1 = \frac{dy}{dx}(5y^4 + 6y^2 + 1)$

f^{-1}

$5y^4 \ldots$

$5y^4 \ldots$

$\frac{dy}{dx} = \frac{1}{5y^4 + 6y^2 + 1} = \frac{1}{5+6+1} = \frac{1}{12}$