4-4 Complex Numbers

$\sqrt{27}$

$3 \quad 9$

$3 \cdot 3$

$3\sqrt{3}$

1 Pure Imaginary Numbers In your math studies so far, you have worked with real numbers. Equations like the one above led mathematicians to define imaginary numbers. The **imaginary unit i** is defined to be $i^2 = -1$. The number i is the principal square root of -1; that is, $i = \sqrt{-1}$.

Numbers of the form $6i$, $-2i$, and $i\sqrt{3}$ are called **pure imaginary numbers**. Pure imaginary numbers are square roots of negative real numbers. For any positive real number b, $\sqrt{-b^2} = \sqrt{b^2} \cdot \sqrt{-1}$ or bi.

PT

Example 1 Square Roots of Negative Numbers

Simplify.

a. $\sqrt{-27}$ $= \sqrt{-1} \cdot \sqrt{27}$

$$\sqrt{-27} = \sqrt{-1 \cdot 3^2 \cdot 3}$$
$$= \sqrt{-1} \cdot \sqrt{3^2} \cdot \sqrt{3}$$
$$= i \cdot 3 \cdot \sqrt{3} \text{ or } 3i\sqrt{3}$$

b. $\sqrt{-216}$

$$\sqrt{-216} = \sqrt{-1 \cdot 6^2 \cdot 6}$$
$$= \sqrt{-1} \cdot \sqrt{6^2} \cdot \sqrt{6}$$
$$= i \cdot 6 \cdot \sqrt{6} \text{ or } 6i\sqrt{6}$$

$\sqrt{-9} = 3i$

$\sqrt{-81}$

$= \sqrt{-1} \cdot \sqrt{81}$

$9i$

Example 2 Products of Pure Imaginary Numbers

Simplify.

a. $-5i \cdot 3i$

$$\begin{aligned} -5i \cdot 3i &= -15i^2 && \text{Multiply.} \\ &= -15(-1) && i^2 = -1 \\ &= 15 && \text{Simplify.} \end{aligned}$$

b. $\sqrt{-6} \cdot \sqrt{-15}$

$$\begin{aligned} \sqrt{-6} \cdot \sqrt{-15} &= i\sqrt{6} \cdot i\sqrt{15} && i = \sqrt{-1} \\ &= i^2\sqrt{90} && \text{Multiply.} \\ &= -1 \cdot \sqrt{9} \cdot \sqrt{10} && \text{Simplify.} \\ &= -3\sqrt{10} && \text{Multiply.} \end{aligned}$$

$i = \sqrt{-1}$

$i^2 = i \cdot i = \sqrt{-1} \cdot \sqrt{-1}$

$= (\sqrt{-1})^2$

$i^2 = -1$

Examples 1–2 **Simplify.**

$16 \cdot 2$

1. $\sqrt{-81}$ **$9i$**
2. $\sqrt{-32}$ **$4i\sqrt{2}$**
3. $(4i)(-3i)$ **12**

$(-12)(-1)$

4. $3\sqrt{-24} \cdot 2\sqrt{-18}$ **$-72\sqrt{3}$**
5. i^{40} **1**
6. i^{63} **$-i$**

$i \cdot i\,(3\sqrt{24})(2\sqrt{18})$

$(i^2)(\sqrt{432})$

$-1\ ($

4 $\quad -24 \times -18 = \sqrt{432}$

$= 12\sqrt{3}$

5. i^{40} 1

$(i^4)^{10}$

i

$i = \sqrt{-1}$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$i^2 \cdot i^2$

$(-1)(-1)$

You can solve some quadratic equations by using the **Square Root Property**. Similar to a difference of squares, the sum of squares can be factored over the complex numbers.

Example 3 Equation with Pure Imaginary Solutions

Solve $x^2 + 64 = 0$.

Method 1 Square Root Property

$$x^2 + 64 = 0$$
$$x^2 = -64$$
$$x = \pm\sqrt{-64}$$
$$x = \pm 8i$$

"No real solution."

Method 2 Factoring

$$x^2 + 64 = 0$$
$$x^2 + 8^2 = 0$$
$$x^2 - (-8^2) = 0$$
$$(x + 8i)(x - 8i) = 0$$
$$(x + 8i) = 0 \text{ or } (x - 8i) = 0$$
$$x = -8i \qquad x = 8i$$

$i^{63} = (i^4)^{15} \cdot (i^3) = i^3 = -i$

$i^2 \cdot i$

(-1)

Example 3 **Solve each equation.**

7. $4x^2 + 32 = 0$ $\pm 2i\sqrt{2}$

8. $x^2 + 1 = 0$ $\pm i$

$4x^2 + 32 = 0$

$-32 \quad -32$

$\frac{4x^2}{4} = \frac{-32}{4}$

$\sqrt{x^2} = \sqrt{-8}$

2, 4, 2, 2, 2, 2

KeyConcept Complex Numbers

"imaginary"

real → $(a + bi)$

Words	A complex number is any number that can be written in the form $a + bi$, where a and b are real numbers and i is the imaginary unit. a is called the real part, and b is called the imaginary part.	
Examples	$5 + 2i$	$1 - 3i = 1 + (-3)i$

Example 4 Equate Complex Numbers

Find the values of x and y that make $3x - 5 + (y - 3)i = 7 + 6i$ true.

Set the real parts equal to each other and the imaginary parts equal to each other.

$3x - 5 = 7$	Real parts	$y - 3 = 6$	Imaginary parts
$3x = 12$	Add 5 to each side.	$y = 9$	Add 3 to each side.
$x = 4$	Divide each side by 3.		

Example 4 **Find the values of a and b that make each equation true.**

9. $3a + (4b + 2)i = 9 - 6i$ **3, −2**

10. $4b - 5 + (-a - 3)i = 7 - 8i$ **5, 3**

$\frac{3a}{3} = \frac{9}{3}$

$a = 3$

$4b + 2 = -6$
$-2 \quad -2$
$\frac{4b}{4} = \frac{-8}{4}$

$b = -2$

Example 5 Add and Subtract Complex Numbers

Simplify.

a. $(5 - 7i) + (2 + 4i)$

$(5 - 7i) + (2 + 4i) = (5 + 2) + (-7 + 4)i$ — Commutative and Associative Properties

$= 7 - 3i$ — Simplify.

b. $(4 - 8i) - (3 - 6i)$

$(4 - 8i) - (3 - 6i) = (4 - 3) + [-8 - (-6)]i$ — Commutative and Associative Properties

$= 1 - 2i$ — Simplify.

Simplify.

11. $(-1 + 5i) + (-2 - 3i)$ $-3 + 2i$

12. $(7 + 4i) - (1 + 2i)$ $6 + 2i$

Example 6

17. **ELECTRICITY** The current in one part of a series circuit is $5 - 3j$ amps. The current in another part of the circuit is $7 + 9j$ amps. Add these complex numbers to find the total current in the circuit. $12 + 6j$ amps

Real-World Example 6 Multiply Complex Numbers

ELECTRICITY **In an AC circuit, the voltage V, current C, and impedance I are related by the formula $V = C \cdot I$. Find the voltage in a circuit with current $2 + 4j$ amps and impedance $9 - 3j$ ohms.**

$V = C \cdot I$	Electricity formula
$= (2 + 4j) \cdot (9 - 3j)$	$C = 2 + 4j$ and $I = 9 - 3j$
$= 2(9) + 2(-3j) + 4j(9) + 4j(-3j)$	FOIL Method
$= 18 - 6j + 36j - 12j^2$	Multiply.
$= 18 + 30j - 12(-1)$	$j^2 = -1$
$= 30 + 30j$	Add.

$i^2 = -1$

The voltage is $30 + 30j$ volts.

13. $(6 - 8i)(9 + 2i)$ **$70 - 60i$**

$54 + 12i - 72i - 16i^2$

$54 - 60i + 16$

14. $(3 + 2i)(-2 + 4i)$ **$-14 + 8i$**

Example 7 Divide Complex Numbers

Simplify.

a. $\dfrac{2i}{3+6i}$

$\dfrac{2i}{3+6i} = \dfrac{2i}{3+6i} \cdot \dfrac{3-6i}{3-6i}$ — $3+6i$ and $3-6i$ are complex conjugates.

$= \dfrac{6i-12i^2}{9-36i^2}$ — Multiply.

$= \dfrac{6i-12(-1)}{9-36(-1)}$ — $i^2 = -1$

$= \dfrac{6i+12}{45}$ — Simplify.

$= \dfrac{4}{15} + \dfrac{2}{15}i$ — $a + bi$ form

b. $\dfrac{4+i}{5i}$

$\dfrac{4+i}{5i} = \dfrac{4+i}{5i} \cdot \dfrac{i}{i}$ — Multiply by $\frac{i}{i}$.

$= \dfrac{4i+i^2}{5i^2}$ — Multiply.

$= \dfrac{4i-1}{-5}$ — $i^2 = -1$

$= \dfrac{1}{5} - \dfrac{4}{5}i$ — $a + bi$ form

15. $\dfrac{3-i}{4+2i}$ $\quad \dfrac{1}{2} - \dfrac{1}{2}i$

16. $\dfrac{2+i}{5+6i}$ $\quad \dfrac{16}{61} - \dfrac{7}{61}i$

$-8i + 8i$

$12 - 6i - 4i + 2i^2$

$12 - 10i + 2(-1)$

15 $\dfrac{(3-i)(4-2i)}{(4+2i)(4-2i)} = \dfrac{10-10i}{16-4i^2}$

$= \dfrac{10-10i}{16-4(-1)} = \dfrac{10-10i}{16+4}$

$\dfrac{10-10i}{20} = \dfrac{10}{20} - \dfrac{10i}{20}$

$= 20$

Practice and Problem Solving

Extra Practice

Examples 1–2 CCSS **STRUCTURE** Simplify.

18. $\sqrt{-121}$ **$11i$**

19. $\sqrt{-169}$ **$13i$**

20. $\sqrt{-100}$ **$10i$**

21. $\sqrt{-81}$ **$9i$**

22. $(-3i)(-7i)(2i)$ **$-42i$**

23. $4i(-6i)^2$ **$-144i$**

24. i^{11} **$-i$**

25. i^{25} **i**

26. $(10 - 7i) + (6 + 9i)$ **$16 + 2i$**

27. $(-3 + i) + (-4 - i)$ **-7**

28. $(12 + 5i) - (9 - 2i)$ **$3 + 7i$**

29. $(11 - 8i) - (2 - 8i)$ **9**

30. $(1 + 2i)(1 - 2i)$ **5**

31. $(3 + 5i)(5 - 3i)$ **$30 + 16i$**

32. $(4 - i)(6 - 6i)$ **$18 - 30i$**

33. $\frac{2i}{1 + i}$ **$1 + i$**

34. $\frac{5}{2 + 4i}$ **$\frac{1}{2} - i$**

35. $\frac{5 + i}{3i}$ **$\frac{1}{3} - \frac{5}{3}i$**

Example 3 Solve each equation.

36. $4x^2 + 4 = 0$ **$\pm i$**

37. $3x^2 + 48 = 0$ **$\pm 4i$**

38. $2x^2 + 50 = 0$ **$\pm 5i$**

39. $2x^2 + 10 = 0$ **$\pm i\sqrt{5}$**

40. $6x^2 + 108 = 0$ **$\pm 3i\sqrt{2}$**

41. $8x^2 + 128 = 0$ **$\pm 4i$**

Example 4 Find the values of x and y that make each equation true.

42. $9 + 12i = 3x + 4yi$ **$3, 3$**

43. $x + 1 + 2yi = 3 - 6i$ **$2, -3$**

44. $2x + 7 + (3 - y)i = -4 + 6i$ **$-\frac{11}{2}, -3$**

45. $5 + y + (3x - 7)i = 9 - 3i$ **$\frac{4}{3}, 4$**

46. $a + 3b + (3a - b)i = 6 + 6i$ **$\frac{12}{5}, \frac{6}{5}$**

47. $(2a - 4b)i + a + 5b = 15 + 58i$ **$25, -2$**