$$\boxed{24} \quad y = (x^2 - 1)^{-1}$$

$$y' = -(x^2 - 1)^{-2} \cdot 2x$$

$$x = 0 \ldots$$

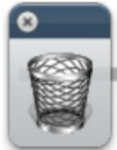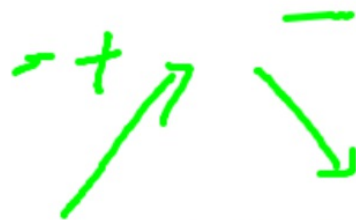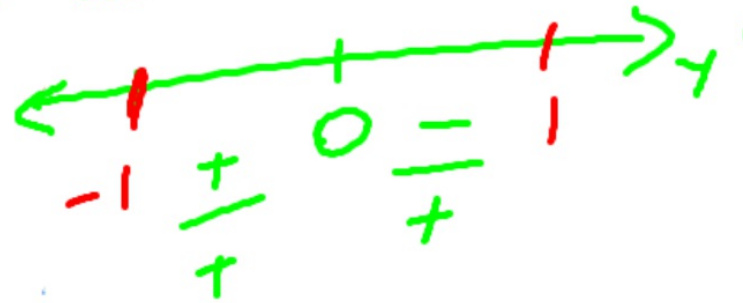$$y' = \frac{-2x}{(x^2 - 1)^2} = 0$$

und. @ $x = -1, 1$

24. $y = \dfrac{1}{x^2 - 1}$ Local max at $(0, -1)$

value 0
$-1, 1$

V.A @ $-1, 1$

test $x = -.5 \quad x = .5$

$$\frac{-1 \quad + \quad 0 \quad = \quad 1}{+ \qquad +}$$

$-+$

$-$

9:21 AM
11/12/201

4.2 The Mean Value Theorem

THEOREM 3 Mean Value Theorem for Derivatives

If $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b), then there is at least one point c in (a, b) at which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Tangent parallel to chord

Slope $f'(c)$

B

Slope $\dfrac{f(b) - f(a)}{b - a}$

A
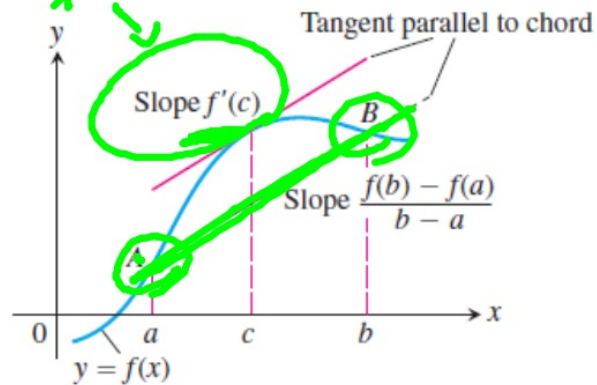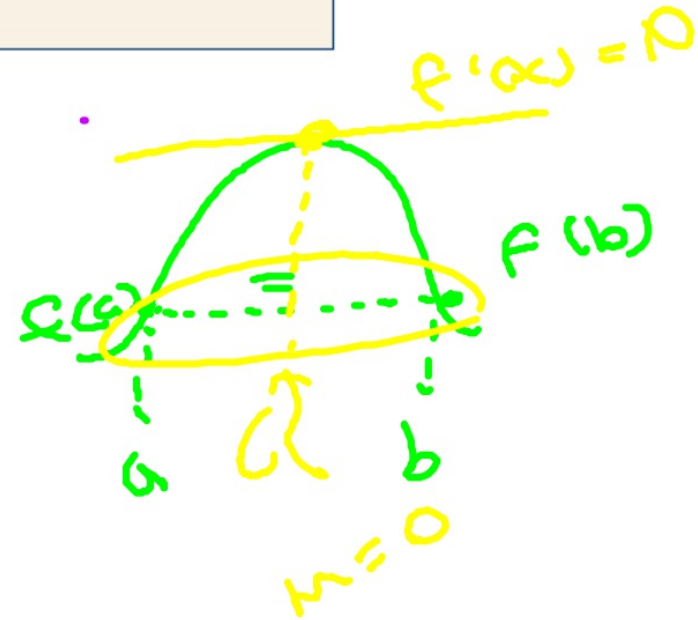
y

0 a c b x

$y = f(x)$

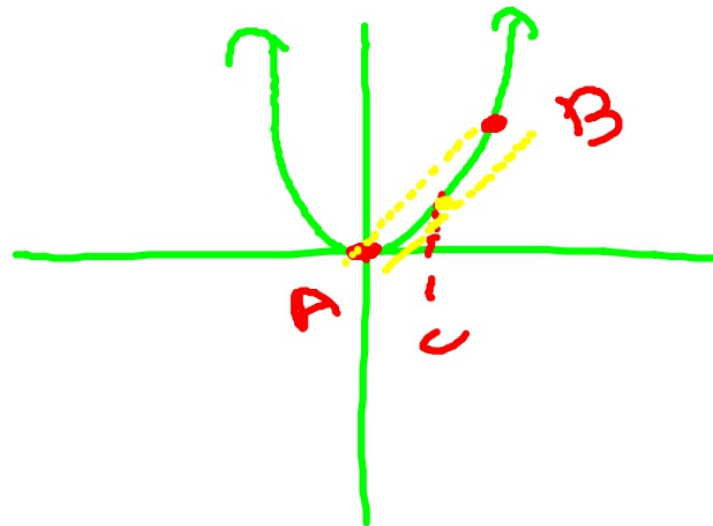**Figure 4.10** Figure for the Mean Value Theorem.

$$c = 1.$$

nterpret The tangent line to $f(x) = x^2$ at $x = 1$ has slope 2 and is parallel to t

hord joining $A(0, 0)$ and $B(2, 4)$ (Figure 4.12).

Now try Exer

continued

THEOREM 3 Mean Value Theorem for Derivatives

If $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b), then there is at least one point c in (a, b) at which
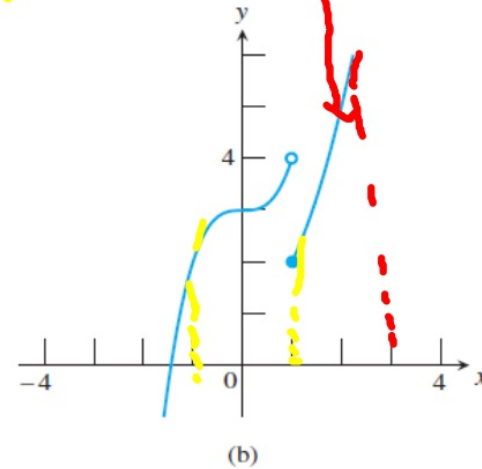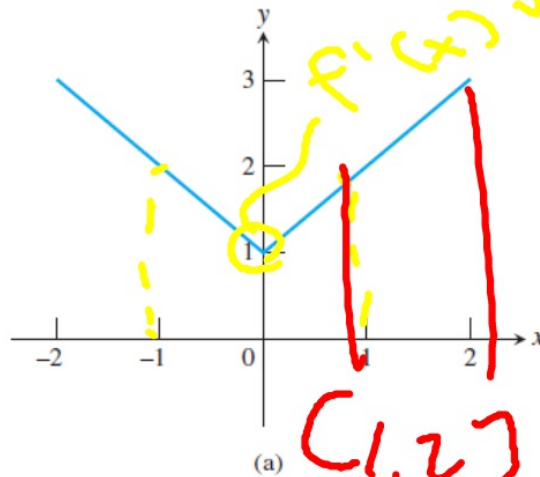
$$f(b) - f(a)$$

EXAMPLE 2 Exploring the Mean Value Theorem

Explain why each of the following functions fails to satisfy the conditions of the Mean Value Theorem on the interval $[-1, 1]$.

(a) $f(x) = \sqrt{x^2 + 1}$

(b) $f(x) = \begin{cases} x^3 + 3 & \text{for } x < 1 \\ x^2 + 1 & \text{for } x \geq 1 \end{cases}$

SOLUTION

$[-1, 2\frac{1}{2}]$

f'(x) und

(a) $[-1, 2]$

(b)

THEOREM 3 Mean Value Theorem for Derivatives

If $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b), then there is at least one point c in (a, b) at which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In Exercises 1–8, **(a)** state whether or not the function satisfies the hypotheses of the Mean Value Theorem on the given interval, and **(b)** if it does, find each value of c in the interval (a, b) that satisfies the equation

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

1. $f(x) = x^2 + 2x - 1$ on $[0, 1]$

2. $f(x) = x^{2/3}$ on $[0, 1]$

3. $f(x) = x^{1/3}$ on $[-1, 1]$ No. There is a vertical tangent at $x = 0$.

4. $f(x) = |x - 1|$ on $[0, 4]$ No. There is a corner at $x = 1$.

5. $f(x) = \sin^{-1}x$ on $[-1, 1]$

6. $f(x) = \ln(x - 1)$ on $[2, 4]$

8. $f(x) = \begin{cases} \sin^{-1} x, & -1 \le x < 1 \\ x/2 + 1, & 1 \le x \le 3 \end{cases}$ on $[-1, 3]$

(handwritten work)

a, b on $[0, 1]$

$f(c) = c^2 + 2c - 1$

$f'(c) = 2c + 2$

$f(1)$

$f(b) = 1 + 2 - 1 = 2$ $f(b)$

$f(a) = f(0) = 0 + 0 - 1 = -1$ $f(a)$

$2c + 2 = \dfrac{2 - (-1)}{1 - 0}$

$2c + 2 = 3$

$2c = 1$

$c = \dfrac{1}{2}$

$\ln(x-1)$ on $[2,4]$

a b

✓ continuous at $[2,4]$
✓ diff. at $(2,4)$

$$\frac{d}{dx}[\ln(u)] = \frac{1}{u} \cdot u'$$

$f(a) = \ln 1$
$f(b) = \ln 3$
$f(4) =$
$\ln(4-1)$
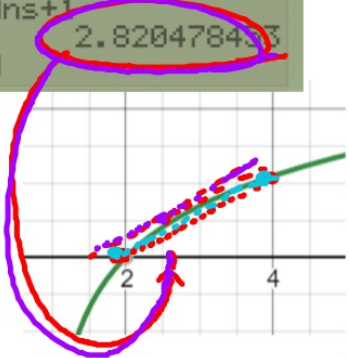
$$f'(c) = \frac{1}{c-1}$$

$u = x-1$
$u' = 1$

$$\frac{f(b) - f(a)}{b-a} = \frac{\ln 3 - \ln 1}{2}$$

$$\frac{1}{c-1} = .549$$

$$1 = (c-1)(.549)$$

$$\frac{1}{.549} = \frac{(c-1)(.549)}{.549}$$

8. $f(x) = \begin{cases} \sin^{-1} x, & -1 \le x < 1 \\ x/2 + 1, & 1 \le x \le 3 \end{cases}$ on $[-1, 3]$
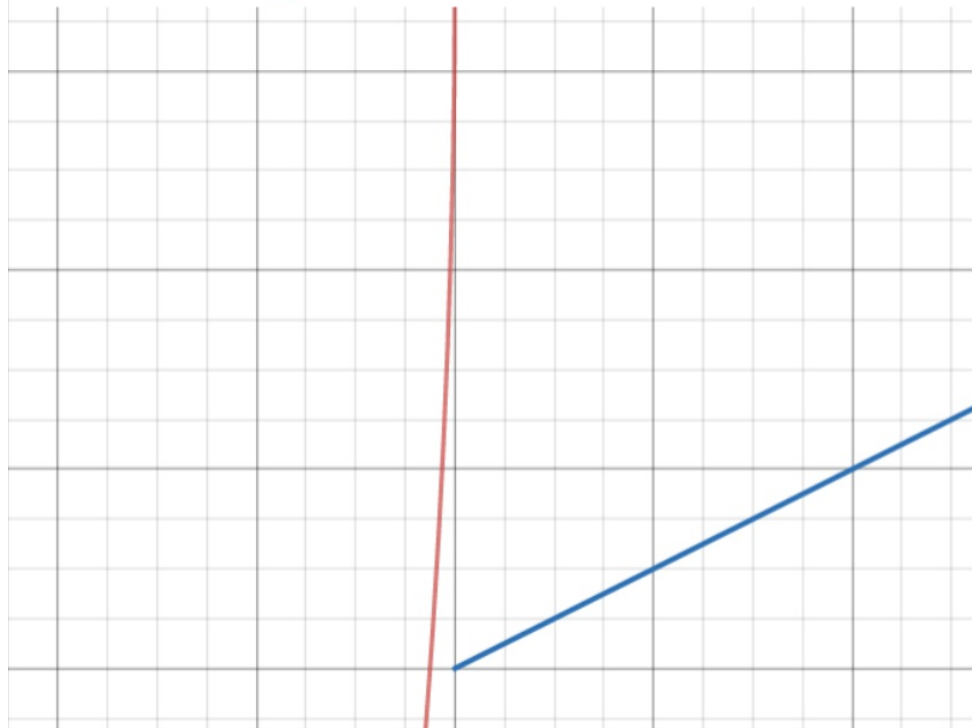
No. The split function is discontinuous at $x = 1$.

In Exercises 1–8, (a) state whether or not the function satisfies the
hypotheses of the Mean Value Theorem on the given interval, and
(b) if it does, find each value of c in the interval (a, b) that satisfies
the equation

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

8. $f(x) = \begin{cases} \sin^{-1} x, & -1 \le x < 1 \\ x/2 + 1, & 1 \le x \le 3 \end{cases}$ on $[-1, 3]$

No. The split function is discontinuous at $x = 1$.

1. $f(x) = x^2 + 2x - 1$ on $[0, 1]$

2. $f(x) = x^{2/3}$ on $[0, 1]$

3. $f(x) = x^{1/3}$ on $[-1, 1]$

4. $f(x) = |x - 1|$ on $[0, 4]$ No. T

5. $f(x) = \sin^{-1} x$ on $[-1, 1]$

6. $f(x) = \ln(x - 1)$ on $[2, 4]$

(1) continuous? ✓
differentiable? ✓

$[0, 1]$

$f'(x) = 2x + 2$

$f(1) = 1^2 + 2(1) - 1$
$= 1 + 2 - 1$
$= 2$

$\frac{f(1) - f(0)}{1 - 0}$

$f(1) = 2$
$f(0) = -1$

$f(0) = 0^2 + 2(0) - 1$
$= -1$

$2c + 2 = \frac{2 - (-1)}{1 - 0}$

$2c + 2 = \frac{3}{1}$

$2c + 2 = 3$

$2c = 1$

$c = \frac{1}{2}$

EXAMPLE 3 Applying the Mean Value Theorem

Let $f(x) = \sqrt{1 - x^2}$, $A = (-1, f(-1))$, and $B = (1, f(1))$. Find a tangent to f in the interval $(-1, 1)$ that is parallel to the secant AB.

SOLUTION

The function f (Figure 4.14) is continuous on the interval $[-1, 1]$ and

$$f'(x) = \frac{-x}{\sqrt{1 - x^2}}$$

is defined on the interval $(-1, 1)$. The function is not differentiable at $x = -1$ and $x = 1$, but it does not need to be for the theorem to apply. Since $f(-1) = f(1) = 0$, the tangent we are looking for is horizontal. We find that $f' = 0$ at $x = 0$, where the graph has the horizontal tangent $y = 1$.

Now try Exercise 9.

COROLLARY 2   Functions with $f' = 0$ are Constant

If $f'(x) = 0$ at each point of an interval I, then there is a constant C for which $f(x) = C$ for all x in I.

In Exercises 9 and 10, the interval $a \le x \le b$ is given. Let $A = (a, f(a))$ and $B = (b, f(b))$. Write an equation for

(a) the secant line AB.

(b) a tangent line to f in the interval (a, b) that is parallel to AB.

9. $f(x) = x + \dfrac{1}{x}$, $0.5 \le x \le 2$ (a) $y = \dfrac{5}{2}$ (b) $y = 2$

10. $f(x) = \sqrt{x - 1}$, $1 \le x \le 3$

$$f(b) = f(2) = 2 + \frac{1}{2}$$

$$f(a) = f(.5) = .5 + 2$$

$$f(b) - f(a) =$$

$$\frac{f(b) - f(a)}{b - a} = ?$$

COROLLARY 1 Increasing and Decreasing Functions

Let f be continuous on $[a, b]$ and differentiable on (a, b).

1. If $f' > 0$ at each point of (a, b), then f increases on $[a, b]$.

2. If $f' < 0$ at each point of (a, b), then f decreases on $[a, b]$.

EXAMPLE 6 Determining Where Graphs Rise or Fall

Where is the function $f(x) = x^3 - 4x$ increasing and where is it decreasing?

SOLUTION

Solve Graphically The graph of f in Figure 4.17 suggests that f is increasing from $-\infty$ to the x-coordinate of the local maximum, decreasing between the two local extrema, and increasing again from the x-coordinate of the local minimum to ∞. This information is supported by the superimposed graph of $f'(x) = 3x^2 - 4$.

Confirm Analytically The function is increasing where $f'(x) > 0$.

$$3x^2 - 4 > 0$$

$$x^2 > \frac{4}{3}$$

$$x < -\sqrt{\frac{4}{3}} \quad \text{or} \quad x > \sqrt{\frac{4}{3}}$$

The function is decreasing where $f'(x) < 0$.

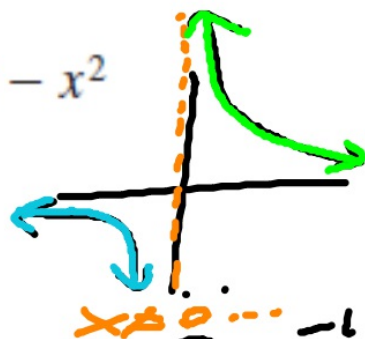$$3x^2 - 4 < 0$$

$$x^2 < \frac{4}{3}$$

$$-\sqrt{\frac{4}{3}} < x < \sqrt{\frac{4}{3}}$$

skip
#11-14!

In Exercises 15–22, use analytic methods to find (a) the local extrema, (b) the intervals on which the function is increasing, and (c) the intervals on which the function is decreasing.

15. $f(x) = 5x - x^2$

16. $g(x) = x^2 - x - 12$

17. $h(x) = \dfrac{2}{x}$

18. $k(x) = \dfrac{1}{x^2}$

First Derivative Test...

$x = 0$ ...

(17) $h(x) = 2x^{-1}$

$h'(x) = -2x^{-2}$

$h'(x) = \dfrac{-2}{x^2}$

Test $x = -1$

Test $x = 1$

h'

$\dfrac{-2}{(-1)^2} = \dfrac{-2}{1} = -2$

-2

Decreasing

Decreasing

In Exercises 15–22, use analytic methods to find (a) the local extrema, (b) the intervals on which the function is increasing, and (c) the intervals on which the function is decreasing.

15. $f(x) = 5x - x^2$ See page 204.

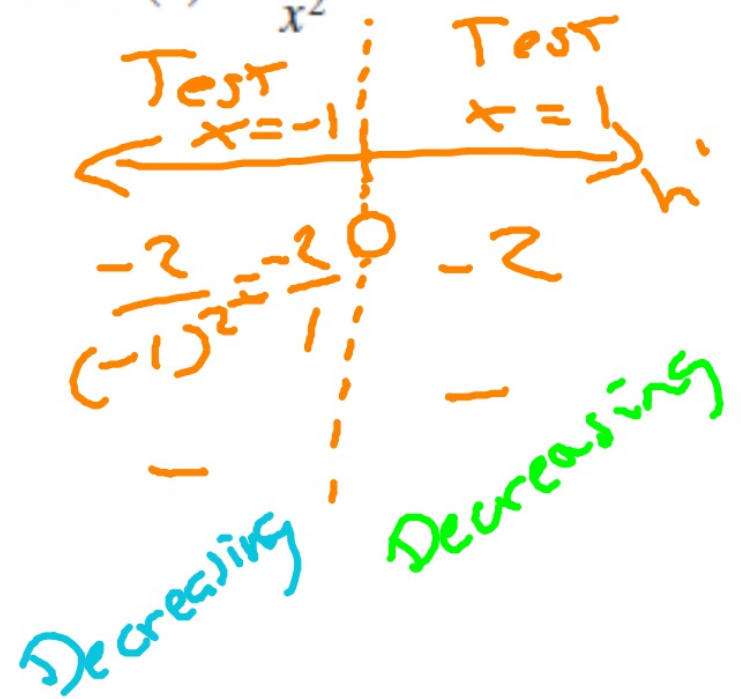16. $g(x) = x^2 - x - 12$ See page 204.

17. $h(x) = \dfrac{2}{x}$ See page 204.

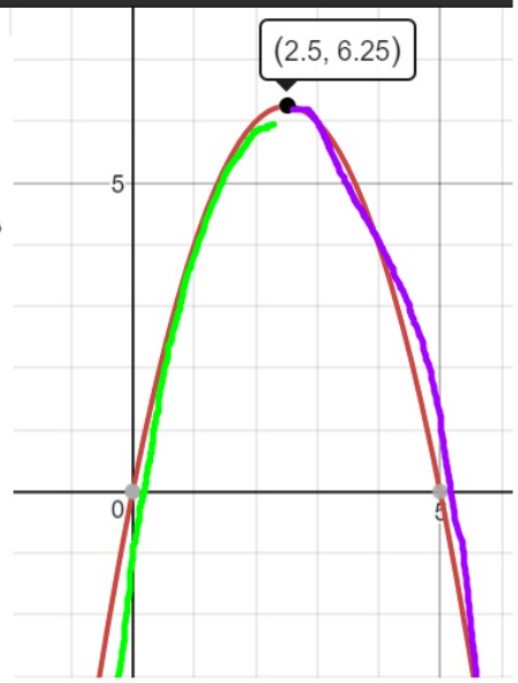18. $k(x) = \dfrac{1}{x^2}$ See page 204.

(15) $f'(x) = 5 - 2x = 0$

$-5 \qquad -5$

$\dfrac{-2x = -5}{-2} \quad \dfrac{}{-2}$

$x = \dfrac{5}{2}$

Test $x = 1$ Test $x = 3$

$5 - 2(1) \quad \dfrac{5}{2} \quad 5 - 2(3)$

$+ \qquad \qquad -$

increasing   *decreasing*

$5x - x^2$

O
S

Test
x=1

Test
x=3

5-2(1) $\frac{5}{2}$ 5-2(3)

$+$ 2 $-$

increasing

decreasing

(2.5, 6.25)

5

0   5

In Exercises 23–28, find (a) the local extrema, (b) the intervals on which the function is increasing, and (c) the intervals on which the function is decreasing.
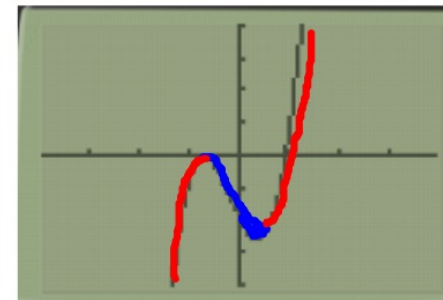
23. $f(x) = x\sqrt{4 - x}$

24. $g(x) = x^{1/3}(x + 8)$

25. $h(x) = \dfrac{-x}{x^2 + 4}$

26. $k(x) = \dfrac{x}{x^2 - 4}$
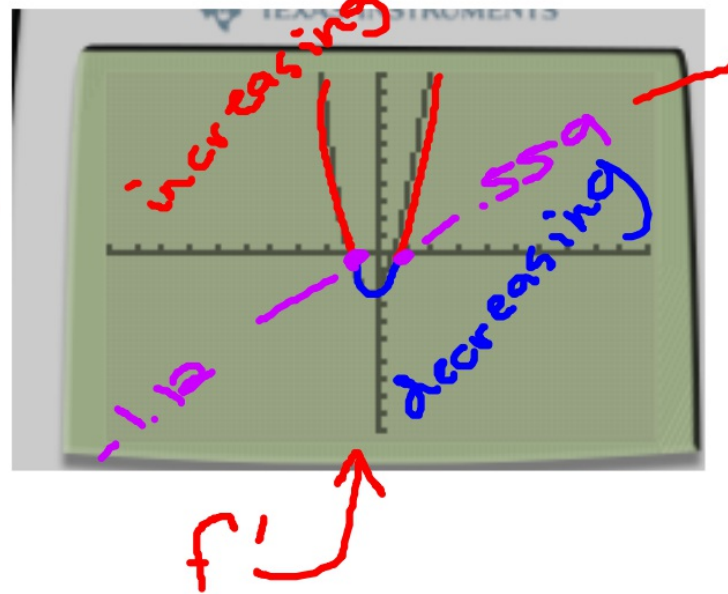
27. $f(x) = x^3 - 2x - 2\cos x$

28. $g(x) = 2x + \cos x$

graphed it ↑↑f

27

Y_1

$f'(x) = 3x^2 - 2 + 2\sin x = 0$

increasing:

$(-\infty, -1.12) \cup (.559, \infty)$

decreasing:

$(-1.12, .559)$

increasing

-1.12

.559

decreasing

f'

In Exercises 23–28, find **(a)** the local extrema, **(b)** the intervals on which the function is increasing, and **(c)** the intervals on which the function is decreasing.

24. (a) Local min at $\approx (-2, -7.56)$

23. $f(x) = x\sqrt{4-x}$

24. $g(x) = x^{1/3}(x+8)$ **(b)** On $[-2, \infty)$
(c) On $(-\infty, -2]$

25. $h(x) = \dfrac{-x}{x^2+4}$ ← f ← g

26. $k(x) = \dfrac{x}{x^2-4}$

27. $f(x) = x^3 - 2x - 2\cos x$

28. $g(x) = 2x + \cos x$
(a) None **(b)** On $(-\infty, \infty)$ **(c)** None

(25)

$f = -x \qquad g = x^2 + 4$

$f' = -1 \qquad g' = 2x$

$h'(x) = \dfrac{(x^2+4)(-1) - (-x)(2x)}{(x^2+4)^2}$

$h'(x) = \dfrac{x^2 - 4}{(x^2+4)^2} = 0$

$x^2 - 4 = 0$

$(x+2)(x-2) = 0$

$x = -2, 2$

Increasing!
$(-\infty, -2) \cup (2, \infty)$

Decreasing:
$(-2, 2)$

Test $x = -3$ Test $x = 1$ Test $x = 3$

$\dfrac{+}{9-4} \dfrac{-}{2} \dfrac{-}{1-4} \quad 2 \dfrac{a \pm 4}{+}$

Local max   Local min

max ⑨ $x = -2$

min ⑨ $x = 2$

Think of deriving, only in reverse....

$$\frac{d}{dx}\left(\ln|u|\right)$$
$$= \frac{u'}{u}$$

In Exercises 29–34, find all possible functions f with the given derivative.

29. $f'(x) = x$ $\quad \frac{x^2}{2} + C$

30. $f'(x) = 2$ $\quad 2x + C$

31. $f'(x) = 3x^2 - 2x + 1$
$\qquad\qquad x^3 - x^2 + x + C$

32. $f'(x) = \sin x$ $\quad -\cos x + C$

33. $f'(x) = e^x$
$\quad e^x + C$

34. $f'(x) = \dfrac{1}{x-1}$, $\quad x > 1$
$\qquad \ln(x - 1) + C$

EXAMPLE 7 Applying Corollary 3

Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0, 2)$.

SOLUTION

Since f has the same derivative as $g(x) = -\cos x$, we know that $f(x) = -\cos x + C$, for some constant C. To identify C, we use the condition that the graph must pass through $(0, 2)$. This is equivalent to saying that

$$f(0) = 2$$

$$-\cos{(0)} + C = 2 \qquad f(x) = -\cos x + C$$

$$-1 + C = 2$$

$$C = 3.$$

The formula for f is $f(x) = -\cos x + 3$.   ***Now try Exercise 35.***

In Exercises 35–38, find the function with the given derivative whose graph passes through the point P.

35. $f'(x) = -\dfrac{1}{x^2}, \quad x > 0, \quad P(2, 1)$

36. $f'(x) = \dfrac{1}{4x^{3/4}}, \quad P(1, -2)$

37. $f'(x) = \dfrac{1}{x + 2}, \quad x > -2, \quad P(-1, 3)$

38. $f'(x) = 2x + 1 - \cos x, \quad P(0, 3)$

(35) $f'(x) = -x^{-2}$

(37) $f(x) = \ln(x+2) + c$

$f(x) = x^{-1} + C$

$f(x) = \dfrac{1}{x} + \dfrac{1}{2}$

To find C....

$f(2) = 2^{-1} + C = \dfrac{1}{2}$

$\dfrac{1}{2} + C = 1$

$C = \dfrac{1}{2}$

Group Activity In Exercises 39–42, sketch a graph of a differentiable function $y = f(x)$ that has the given properties.

39. (a) local minimum at $(1, 1)$, local maximum at $(3, 3)$

   (b) local minima at $(1, 1)$ and $(3, 3)$

   (c) local maxima at $(1, 1)$ and $(3, 3)$

40. $f(2) = 3$, $f'(2) = 0$, and

   (a) $f'(x) > 0$ for $x < 2$, $f'(x) < 0$ for $x > 2$.

   (b) $f'(x) < 0$ for $x < 2$, $f'(x) > 0$ for $x > 2$.

   (c) $f'(x) < 0$ for $x \ne 2$.

   (d) $f'(x) > 0$ for $x \ne 2$.

41. $f'(-1) = f'(1) = 0$, $f'(x) > 0$ on $(-1, 1)$,
    $f'(x) < 0$ for $x < -1$, $f'(x) > 0$ for $x > 1$.