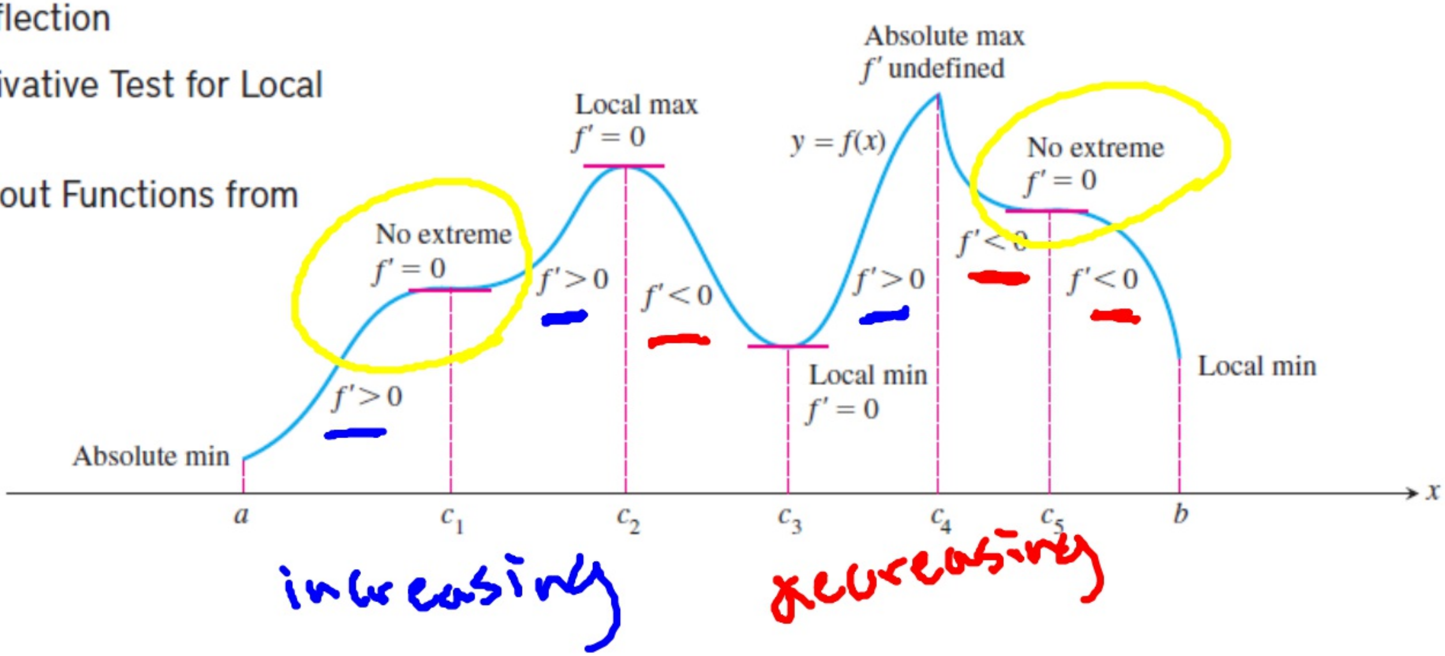
4.3 Connecting f' and f'' with the Graph of f

What you'll learn about

- First Derivative Test for Local Extrema
- Concavity
- Points of Inflection
- Second Derivative Test for Local Extrema
- Learning about Functions from Derivatives

We will use what we know about derivatives to identify extremas;

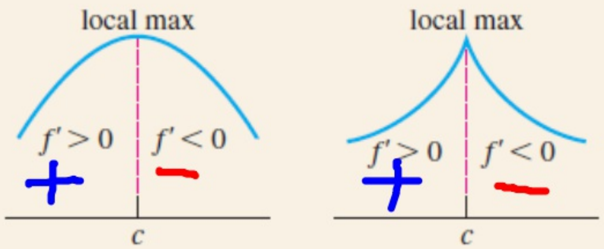


THEOREM 4 First Derivative Test for Local Extrema

The following test applies to a continuous function $f(x)$.

At a critical point c:

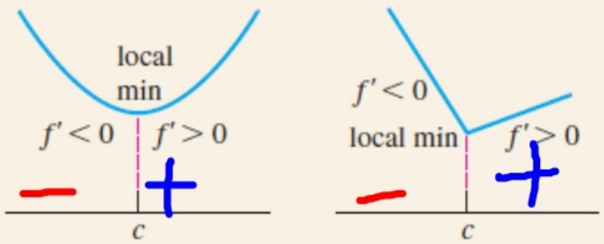
1. If f' changes sign from positive to negative at c ($f' > 0$ for $x < c$ and $f' < 0$ for $x > c$), then f has a local maximum value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

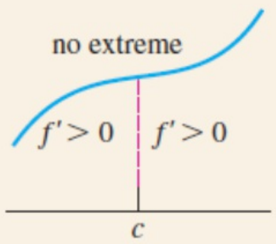
notice how the sign of f' changes around extremas!

2. If f' changes sign from negative to positive at c ($f' < 0$ for $x < c$ and $f' > 0$ for $x > c$), then f has a local minimum value at c.

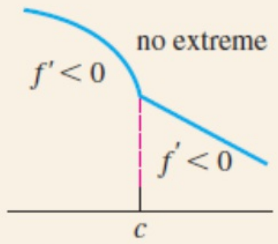


(a) $f'(c) = 0$ (b) $f'(c)$ undefined

3. If f' does not change sign at c (f' has the same sign on both sides of c), then f has no local extreme value at c.



(a) $f'(c) = 0$

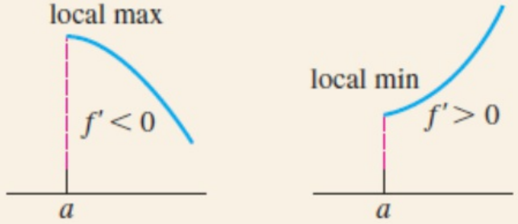


(b) $f'(c)$ undefined

Also, note how extremas don't occur whenever the sign of f' does NOT change...

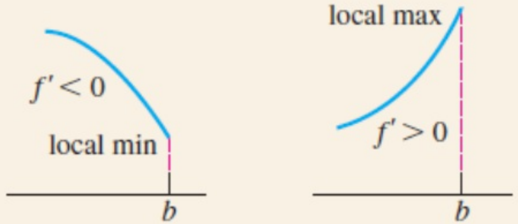
At a left endpoint a:

If $f' < 0$ ($f' > 0$) for $x > a$, then f has a local maximum (minimum) value at a.



At a right endpoint b:

If $f' < 0$ ($f' > 0$) for $x < b$, then f has a local minimum (maximum) value at b.



EXAMPLE 1 Using the First Derivative Test

For each of the following functions, use the First Derivative Test to find the local extreme values. Identify any absolute extrema.

(a) $f(x) = x^3 - 12x - 5$

$f'(x) = 3x^2 - 12$

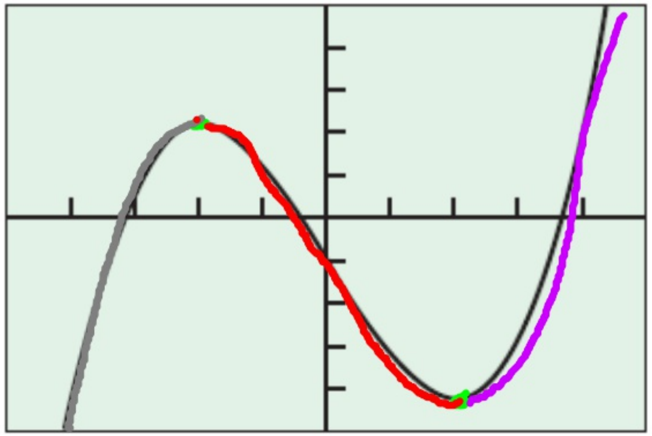
$3x^2 - 12 = 0$

$+12 \quad +12$

$\frac{3x^2}{3} = \frac{12}{3}$

$\sqrt{x^2} = \sqrt{4}$

$x = \pm 2$

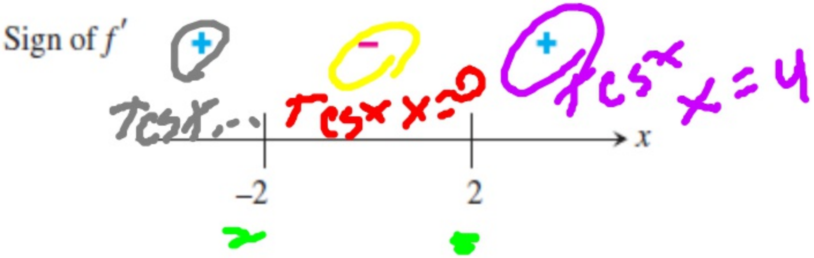


$3(0)^2 - 12$

$= -12$

SOLUTION

(a) Since f is differentiable for all real numbers, the only possible critical points are the zeros of f'. Solving $f'(x) = 3x^2 - 12 = 0$, we find the zeros to be $x = 2$ and $x = -2$. The zeros partition the x-axis into three intervals, as shown below:



$3(4)^2 - 12$

$3(16)$

$48 - 12 = 36$

4. $y = xe^{1/x}$ Local minimum: $(1, e)$

$x=1 \ldots$ $y = (1)e^{1/1}$

$\frac{d}{dx} e^u = e^u \cdot u'$

$u = -x$

$u' = -1$

(4) $y = x \cdot e^{-x}$

$f = x$ $\qquad g = e^{-x}$

$f' = 1$ $\qquad g' = -e^{-x}$

$f' = (1)(e^{-x}) + (-e^{-x})(x)$

GCF: e^{-x}

$f' = e^{-x} - e^{-x} \cdot x = 0$

$e^{-2} = \frac{1}{e^2}$ $+$

$f' = e^{-x}(1-x) = 0$

zero!! $e^{-x} = 0$ $\rightarrow$ $\frac{1}{e^x} = 0$

$1 - x = 0$

$x = 1$ $\qquad x = 2$

$x = 0$

$e^0(1-0) \mid e^{-2}(1-2)$

$+$ $\qquad -$

$x = 1$

Local maxima: $(-\sqrt{8}, 0)$ and $(2, 4)$; Local minimum: $(0, 1)$

5. $y = x\sqrt{8 - x^2}$

local minima: $(-2, -4)$ and $(\sqrt{8}, 0)$;

4 is an absolute maximum and -4 is an absolute minimum.

6. $y = \begin{cases} 3 - x^2, & x < 0 \\ x^2 + 1, & x \geq 0 \end{cases}$

(5) $y = x(8-x^2)^{1/2}$

$f = x \qquad g = (8-x^2)^{1/2}$

$f' = 1 \qquad g' = \frac{1}{2}(8-x^2)^{-1/2}(-2x)$

$x^n \cdot x^m = x^{n+m}$

$\frac{2(8-x^2)^{1/2}}{2(8-x^2)^{1/2}} (8-x^2)^{1/2} + \frac{-2x^2}{2(8-x^2)^{1/2}} = 0$

$\frac{2(8-x^2) + (-2x^2)}{2(8-x^2)^{1/2}} = \frac{16-2x^2-2x^2}{2(8-x^2)^{1/2}}$

$= \frac{16-4x^2}{2(8-x^2)^{1/2}} = \frac{8-2x^2}{(8-x^2)^{1/2}} = 0$

corners/c.p.: $8 - x^2 = 0 \Rightarrow 8 = x^2$, $x = \pm\sqrt{8} = \pm 2\sqrt{2}$

max/min: $8 - 2x^2 = 0$, $8 = 2x^2$, $x^2 = 4$, $x = \pm 2$

Test $x=-3$; Test $x=-2.5$; Test $x=0$; Test 2.5; Test $x=3$

$\sqrt{8-x^2}$

$-2\sqrt{2}$, -2, 2, $2\sqrt{2}$

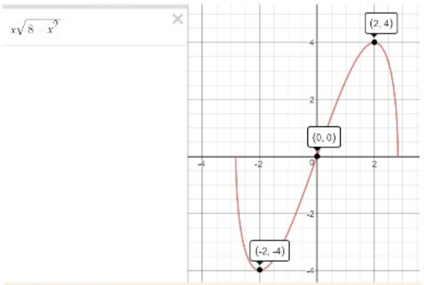
-2.8, 2.8

$-$ $+$ $-$

$\frac{8-12.5}{+}$ $\quad \frac{8}{\sqrt{8}}$

f'

min max

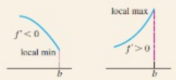


At a left endpoint a:

If $f' < 0$ $(f' > 0)$ for $x > a$, then f has a local maximum (minimum) value at a.

At a right endpoint b:

If $f' < 0$ $(f' > 0)$ for $x < b$, then f has a local minimum (maximum) value at b.



In Exercises 1–6, use the First Derivative Test to determine the local extreme values of the function, and identify any absolute extrema. Support your answers graphically.

1. $y = x^2 - x - 1$ **2.** $y = -2x^3 + 6x^2 - 3$

① $y = x^2 - x - 1$

$y' = 2x - 1 = 0$

$+1 \quad +1$

$2x = 1$

$\frac{2x}{2} = \frac{1}{2}$

$x = \frac{1}{2}$

$y = \left(\frac{1}{2}\right)^2 - \frac{1}{2} - 1$

$= \frac{1}{4} - \frac{1}{2} - 1 = -\frac{1}{4} - 1 = -1\frac{1}{4}$

$\left(\frac{1}{2}, -1\frac{1}{4}\right)$

① Derive, set equal to zero. find x!

Test $x = 0$ Test $x = 1$

f'

$2(0) - 1$ $\quad$ $\frac{1}{2}$ $\quad$ $2(1) - 1$

$-$ $\quad$ $+$

decreases $\quad$ increases

Find it! → min

② Test around crit #

③ Use f to locate extrema

In Exercises 1–6, use the First Derivative Test to determine the local extreme values of the function, and identify any absolute extrema. Support your answers graphically.

1. $y = x^2 - x - 1$

2. $y = -2x^3 + 6x^2 - 3$

3. $y = 2x^4 - 4x^2 + 1$

4. $y = xe^{1/x}$ Local minimum: $(1, e)$

Local maxima: $(-\sqrt{8}, 0)$ and $(2, 4)$;

Local minimum: $(0, 1)$

5. $y = x\sqrt{8 - x^2}$

local minima: $(-2, -4)$ and $(\sqrt{8}, 0)$;

4 is an absolute maximum and -4 is an absolute minimum.

6. $y = \begin{cases} 3 - x^2, & x < 0 \\ x^2 + 1, & x \geq 0 \end{cases}$

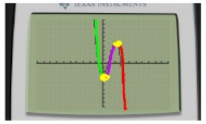
(2) $y' = -6x^2 + 12x = 0 \quad x = 0, 2$

$-6x(x-2) = 0$

Test $x=-1$ Test $x=1$ Test $x=3$

$(+)(-)$ 0 $(-x-)$ 2 $(-x+)$ f'

$-$ min $+$ max $-$



(5) $y = x\sqrt{8-x^2}$

f g

$8 - x^2 > 0$

$8 > x^2$

$x^2 < 8$

$x < \sqrt{8}$ $x > -\sqrt{8}$

$(-\sqrt{8}, \sqrt{8})$

$f = x$ $g = (8-x^2)^{\frac{1}{2}}$

$f' = 1$ $g' = \frac{1}{2}(8-x^2)^{-\frac{1}{2}}(-2x)$

$= -x(8-x^2)^{-\frac{1}{2}}$

$y' = (1)(8-x^2)^{\frac{1}{2}} - x^2(8-x^2)^{-\frac{1}{2}}$

$= \frac{\sqrt{8-x^2}}{1} - \frac{x^2}{\sqrt{8-x^2}} = \frac{8-x^2}{\sqrt{8-x^2}} - \frac{x^2}{\sqrt{8-x^2}}$

$y' = \frac{8-2x^2}{\sqrt{8-x^2}} = 0$

$8 - 2x^2 = 0$

$-8 \quad -8$

$-2x^2 = -8$

$\sqrt{x^2} = \sqrt{4}$

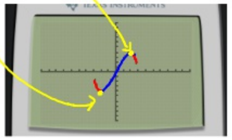
$x = \pm 2$

Test $x=0$

max min y'

$-\sqrt{8}$ $-$ -2 $+$ $+2$ $-$ $+\sqrt{8}$

min max



In Exercises 1–6, use the First Derivative Test to determine the local extreme values of the function, and identify any absolute extrema. Support your answers graphically.

1. $y = x^2 - x - 1$

2. $y = -2x^3 + 6x^2 - 3$

3. $y = 2x^4 - 4x^2 + 1$

4. $y = xe^{1/x}$ Local minimum: $(1, e)$

Local maxima: $(-\sqrt{8}, 0)$ and $(2, 4)$;

Local minimum: $(0, 1)$

5. $y = x\sqrt{8 - x^2}$

6. $y = \begin{cases} 3 - x^2, & x < 0 \\ x^2 + 1, & x \geq 0 \end{cases}$

local minima: $(-2, -4)$ and $(\sqrt{8}, 0)$;
4 is an absolute maximum and -4 is an absolute minimum.

(4) $y = x e^{1/x}$

$f \quad g$

$f = x \quad g = e^{x^{-1}}$

$f' = 1 \quad g' = e^{\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right)$

$y' = e^{\frac{1}{x}} + x \cdot e^{\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right)$

If $x = 1$,

$y' = e^{\frac{1}{x}} - \frac{1}{x} e^{\frac{1}{x}}$

$y' = 0$

$x \neq 0$

$\sqrt{e} - \frac{1}{2}\sqrt{e}$

$x = 2$

$= \frac{1}{2}\sqrt{e}$

$x = \frac{1}{2}$

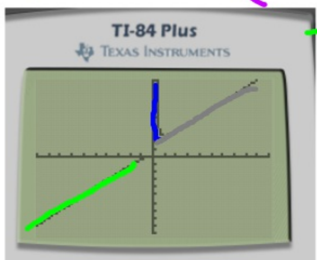
$x = -1 \ldots$

$(+) \; (+)(+)$

$\frac{1}{e} + \frac{1}{e}$

$+ \quad 0 \quad - \quad 1 \quad + \quad y'$

$e^2 - 2e^2 = -e^2$



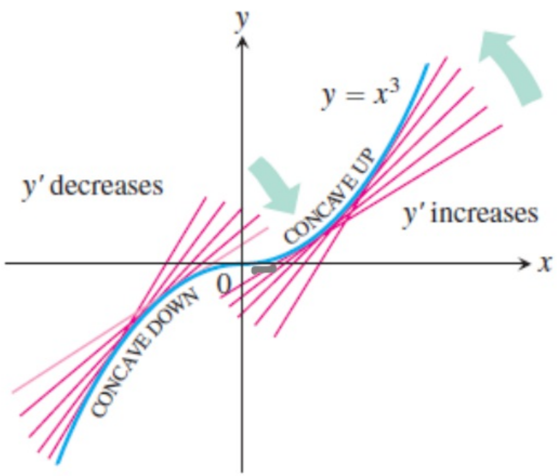
DEFINITION Concavity

The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if y' is increasing on I.

(b) **concave down** on an open interval I if y' is decreasing on I.

As you can see in Figure 4.21, the function $y = x^3$ rises as x increases, but the portions defined on the intervals $(-\infty, 0)$ and $(0, \infty)$ *turn* in different ways.



Derive y *twice*...

Concavity Test

The graph of a twice-differentiable function $y = f(x)$ is

(a) concave up on any interval where $y'' > 0$.

(b) concave down on any interval where $y'' < 0$.

EXAMPLE 2 Determining Concavity

Use the Concavity Test to determine the concavity of the given functions on the given intervals:

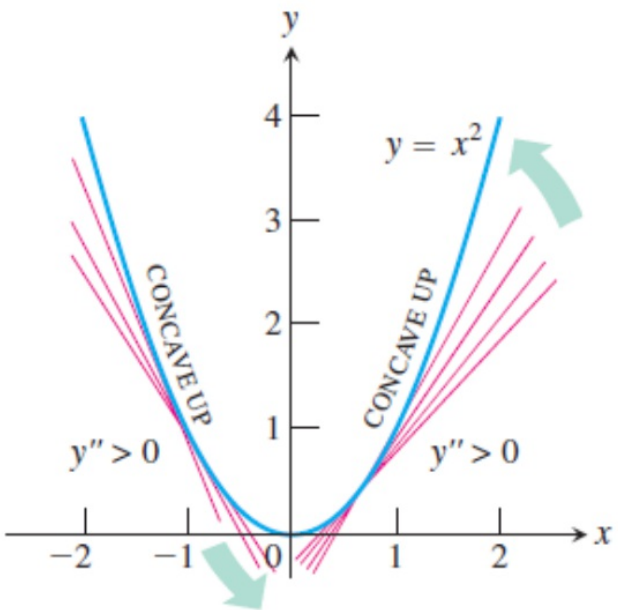
(a) $y = x^2$ on $(3, 10)$ **(b)** $y = 3 + \sin x$ on $(0, 2\pi)$

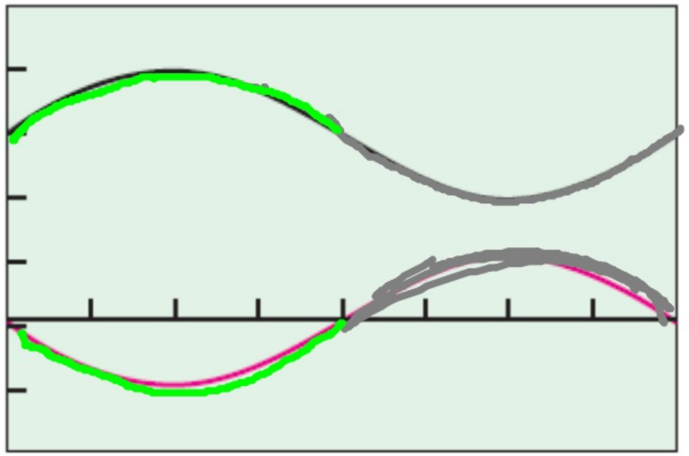
SOLUTION

(a) Since $y'' = 2$ is always positive, the graph of $y = x^2$ is concave up on *any* interval. In particular, it is concave up on $(3, 10)$ (Figure 4.22).

(b) The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$, where $y'' = -\sin x$ is negative. It is concave up on $(\pi, 2\pi)$, where $y'' = -\sin x$ is positive (Figure 4.23).

Now try Exercise 7.





$[0, 2\pi]$ by $[-2, 5]$

In Exercises 7–12, use the Concavity Test to determine the intervals on which the graph of the function is **(a)** concave up and **(b)** concave down.

7. $y = 4x^3 + 21x^2 + 36x - 20$
(a) $(-7/4, \infty)$ **(b)** $(-\infty, -7/4)$

8. $y = -x^4 + 4x^3 - 4x + 1$
(a) $(0, 2)$ **(b)** $(-\infty, 0)$ and $(2, \infty)$

9. $y = 2x^{1/5} + 3$
(a) $(-\infty, 0)$ **(b)** $(0, \infty)$

10. $y = 5 - x^{1/3}$
(a) $(0, \infty)$ **(b)** $(-\infty, 0)$

11. $y = \begin{cases} 2x, & x < 1 \\ 2 - x^2, & x \ge 1 \end{cases}$
(a) None **(b)** $(1, \infty)$

12. $y = e^x, \quad 0 \le x \le 2\pi$
(a) $(0, 2\pi)$ **(b)** None

8

$y' = -4x^3 + 12x^2 - 4$ no una.

$y'' = -12x^2 + 24x$

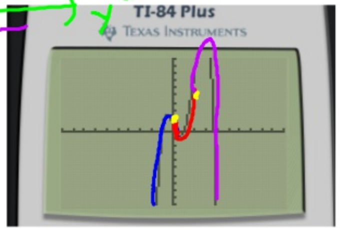
$-12x^2 + 24x = 0$ $x = 0, 2$

$-12x(x-2) = 0$

$x=-1$ $(+)(-)$ $x=1$ $(-)(-)$ $x=3$ $(-)(+)$ y''

$-$ 0 $+$ 2 $-$

use f(x)!



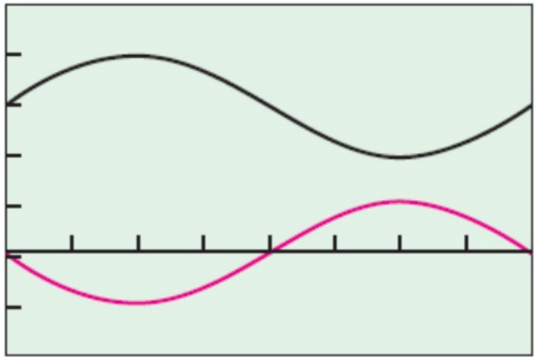
$y = e^x$ $0 \le x \le 2\pi$

$y' = e^x$ $e^0 = 1$

$y'' = e^x$ $e^{2\pi} = +$

This is always positive

$y_1 = 3 + \sin x$, $y_2 = -\sin x$



$[0, 2\pi]$ by $[-2, 5]$

Points of Inflection

The curve $y = 3 + \sin x$ in Example 2 changes concavity at the point $(\pi, 3)$. We call $(\pi, 3)$ a *point of inflection* of the curve.

DEFINITION Point of Inflection

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection.**

A point on a curve where y'' is positive on one side and negative on the other is a point of inflection. At such a point, y'' is either zero (because derivatives have the intermediate value property) or undefined.

To find points of inflection, let f'' = 0 zero!

7. $y = 4x^3 + 21x^2 + 36x - 20$

(a) $(-7/4, \infty)$ (b) $(-\infty, -7/4)$

$y' = 12x^2 + 42x + 36$

$y'' = 24x + 42 = 0$

$-42 \quad -42$

$\frac{24x}{24} = \frac{-42}{24}$

$x = \frac{-7}{4}$

$y'' = 24x + 42$

Test $x = -2$

$-48 + 42$

$-$

Test $x = 0$

$0 + 42$

$+$

$-\frac{7}{4}$

y''

$(-\infty, -\frac{7}{4})$ concave down

concave up $(-\frac{7}{4}, \infty)$

EXAMPLE 3 Finding Points of Inflection

Find all points of inflection of the graph of $y = e^{-x^2}$.

SOLUTION

First we find the second derivative, recalling the Chain and Product Rules:

$$
\begin{aligned}
y &= e^{-x^2} \\
y' &= e^{-x^2} \cdot (-2x) \\
y'' &= e^{-x^2} \cdot (-2x) \cdot (-2x) + e^{-x^2} \cdot (-2) = 0 \\
&= e^{-x^2}(4x^2 - 2) = 0
\end{aligned}
$$

The factor e^{-x^2} is always positive, while the factor $(4x^2 - 2)$ changes sign at $-\sqrt{1/2}$ and at $\sqrt{1/2}$. Since y'' must also change sign at these two numbers, the points of inflection are $(-\sqrt{1/2}, 1/\sqrt{e})$ and $(\sqrt{1/2}, 1/\sqrt{e})$. We confirm our solution graphically by observing the changes of curvature in Figure 4.24.

Now try Exercise 13.

In Exercises 13–20, find all points of inflection of the function.

13. $y = xe^x$

14. $y = x\sqrt{9 - x^2}$

15. $y = \tan^{-1} x$

16. $y = x^3(4 - x)$

17. $y = x^{1/3}(x - 4)$

18. $y = x^{1/2}(x + 3)$

19. $y = \dfrac{x^3 - 2x^2 + x - 1}{x - 2}$

20. $y = \dfrac{x}{x^2 + 1}$

15

$y = \tan^{-1} x$

$y' = \dfrac{1}{1+x^2} = (1+x^2)^{-1}$

$y'' = -(1+x^2)^{-2} \cdot 2x$

$y'' = \dfrac{-2x}{(1+x^2)^2}$

$1 + x^2 = 0$

$x^2 = -1$

$x = \sqrt{-1}$

$= i$

nothing undef.

if $x = 0$

then

$y = \tan^{-1} 0$

$y = 0$

P.O.I.

$(0, 0)$

if $x = 0$,

$y'' = 0$

$x = -1$ $\quad$ $x = 1$

$\dfrac{+}{+} = (+)$ $\quad$ $\dfrac{-}{+} = (-)$

13

$y = \cancel{x}e^x$

$y'' = \cancel{xe^x} \; e^x(2+x) = 0$

ALWAYS "+"!!!

$x = -2$

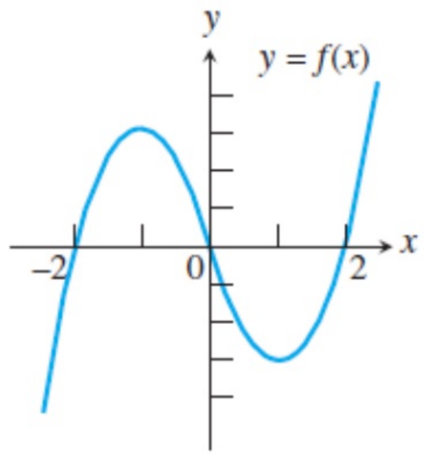
$x = -3$ $\qquad$ $x = 0$

$-$ $\quad$ -2 $\quad$ $+$ $\quad$ y''

$(+)(-)$ $\qquad$ $(+)(+)$

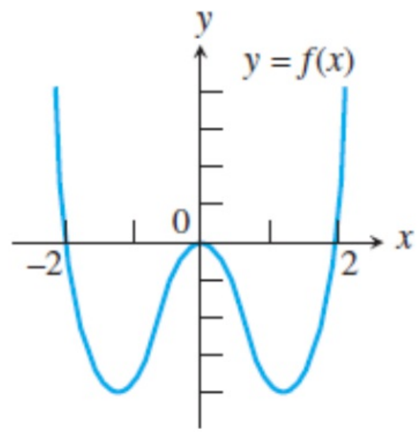
In Exercises 21 and 22, use the graph of the function f to estimate where **(a)** f' and **(b)** f'' are 0, positive, and negative.

21.



(**a**) Zero: $x = \pm 1$;
positive; $(-\infty, -1)$ and $(1, \infty)$;
negative: $(-1, 1)$
(**b**) Zero: $x = 0$;
positive: $(0, \infty)$;
negative: $(-\infty, 0)$

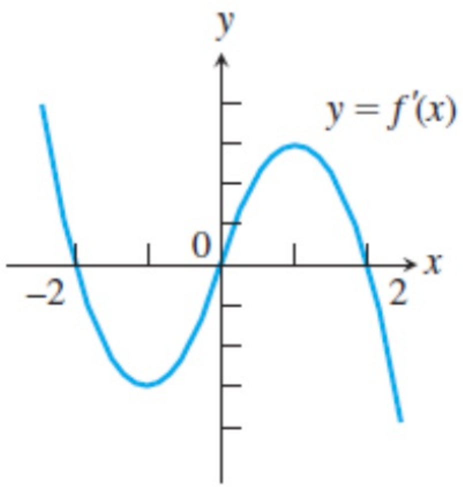
22.



(**a**) Zero: $x \approx 0, \pm 1.25$;
positive: $(-1.25, 0)$ and $(1.25, \infty)$;
negative: $(-\infty, -1.25)$ and $(0, 1.25)$
(**b**) Zero: $x \approx \pm 0.7$;
positive: $(-\infty, -0.7)$ and $(0.7, \infty)$;
negative: $(-0.7, 0.7)$

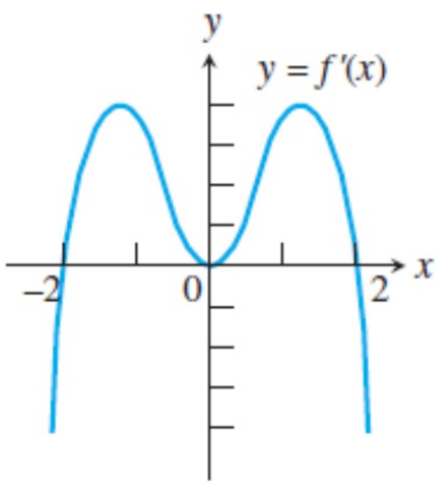
In Exercises 23 and 24, use the graph of the function f' to estimate the intervals on which the function f is (**a**) increasing or (**b**) decreasing. Also, (**c**) estimate the x-coordinates of all local extreme values.

23.



(**a**) $(-\infty, -2]$ and $[0, 2]$
(**b**) $[-2, 0]$ and $[2, \infty)$
(**c**) Local maxima: $x = -2$ and $x = 2$;
local minimum: $x = 0$

24.



(**a**) $[-2, 2]$ (**b**) $(-\infty, -2]$ and $[2, \infty)$
(**c**) Local maximum: $x = 2$;
local minimum: $x = -2$