4.4 Modeling and Optimization

x	$A(x)$
1	5
2	8
3	9
4	8

Suppose I gave you a 12 inch string to make a rectangle.

$$\frac{2x}{2}+\frac{2y}{2}=\frac{12}{2} \leftarrow \text{secondary}$$

$$x+y=6$$
$$-x \qquad -x$$
$$y=6-x$$
$$y=6-3=3$$

$$xy = \text{biggest Area}$$

③ Find extremas (derive, set = to zero)

$$A(x) = -x^2+6x$$
$$A'(x) = -2x+6=0$$
$$x=3$$

② change "primary" into one variable

$$x(6-x) = \text{biggest area}$$
$$-x^2+6x = \text{primary}$$

① find "primary" and "secondary"

$$A(3) = -3^2+6(3)$$
$$= -9+18$$
$$= 9$$
$$A(3) = 9$$

④ Answer the question.

what's the biggest area you can make with it?

Strategy for Solving Max-Min Problems

1. **Understand the Problem** Read the problem carefully. Identify the information you need to solve the problem.
2. **Develop a Mathematical Model of the Problem** Draw pictures and label the parts that are important to the problem. Introduce a variable to represent the quantity to be maximized or minimized. Using that variable, write a function whose extreme value gives the information sought.
3. **Graph the Function** Find the domain of the function. Determine what values of the variable make sense in the problem.
4. **Identify the Critical Points and Endpoints** Find where the derivative is zero or fails to exist.
5. **Solve the Mathematical Model** If unsure of the result, support or confirm your solution with another method.
6. **Interpret the Solution** Translate your mathematical result into the problem setting and decide whether the result makes sense.

In Exercises 1–10, solve the problem analytically. Support your answer graphically.

1. ***Finding Numbers*** The sum of two nonnegative numbers is 20. Find the numbers if

   (a) the sum of their squares is as large as possible; as small as possible.

   (b) one number plus the square root of the other is as large as possible; as small as possible.

$x + y = 20$ $\quad f' = 0$

a) $x^2 + y^2 =$ large as possible

b) $x + \sqrt{y} =$ large as possible

a) $x + y = 20$

$y = 20 - x$

$x^2 + y^2 =$ large

$x^2 + (20 - x)^2 = \#$

$x^2 + 400 - 40x + x^2 = \#$

$400 - 40x + 2x^2 = \#$

$-40 + 4x = 0$

$x = 10 \qquad y = 10$

smallest

3. ***Maximizing Perimeter*** What is the smallest perimeter possible for a rectangle whose area is 16 in^2, and what are its dimensions? Smallest perimeter = 16 in., dimensions are 4 in. by 4 in.

4. ***Finding Area*** Show that among all rectangles with an 8-m perimeter, the one with largest area is a square. See page 232.

3

primary

$2x+2y=$ smallest #

secondary

$xy=16$

$y=\frac{16}{x}$

$2x+2\left(\frac{16}{x}\right)=\#$

$2x+32x^{-1}=\#$

$2-32x^{-2}=0$

$2=\frac{32}{x^2}$

$x^2=\frac{32}{2}$

$x^2=16$

$x=4$

$y=4$

4

secondary

$2x+2y=8$

$xy=$ largest #

primary

$x=4-y$

$(4-y)y=\#$

$4y-y^2=\#$

$4-2y=0$

$4=2y$

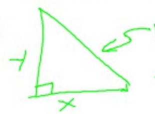
$y=2 \quad x=2$

2. ***Maximizing Area*** What is the largest possible area for a right triangle whose hypotenuse is 5 cm long, and what are its dimensions?

3. ***Maximizing Perimeter*** What is the smallest perimeter possible for a rectangle whose area is 16 in^2, and what are its dimensions? Smallest perimeter = 16 in., dimensions are 4 in. by 4 in.

4. ***Finding Area*** Show that among all rectangles with an 8-m perimeter, the one with largest area is a square. See page 232.

(2)



$h=5$

largest # $\rightarrow A = \frac{1}{2}xy$ ← "primary"

$x^2+y^2=25$ ← "secondary"

$y^2 = 25-x^2$

$y=\sqrt{25-x^2}$

$f=\frac{1}{2}x \quad g=(25-x^2)^{\frac{1}{2}}$

$f'=\frac{1}{2} \quad g'=\frac{1}{2}(25-x^2)^{-\frac{1}{2}}(-2x)$

$\frac{d}{dx}\left[\frac{1}{2}x\left(\sqrt{25-x^2}\right)\right]$

$\left(\frac{1}{2}x\right)\left(\frac{1}{2}(25-x^2)^{-\frac{1}{2}}(-2x)\right)+\frac{1}{2}(25-x^2)^{\frac{1}{2}}=0$

$2\left(\left(\frac{1}{2}x\right)\left(\frac{1}{2}(25-x^2)^{-\frac{1}{2}}(-2x)\right)+\frac{1}{2}(25-x^2)^{\frac{1}{2}}\right)=0$

$\frac{-x^2}{(25-x^2)^{\frac{1}{2}}}+\frac{(25-x^2)^{\frac{1}{2}}(25-x^2)^{\frac{1}{2}}}{(25-x^2)^{\frac{1}{2}}}$

$=\frac{-x^2+(25-x^2)}{(25-x^2)^{\frac{1}{2}}}=\frac{-2x^2+25}{(25-x^2)^{\frac{1}{2}}}=$

$-2x^2+25=0$

$-25 \quad -25$

$-2x^2=-25$

$\frac{-2x^2}{-2}=\frac{-25}{-2}$

$x^2=\frac{25}{2}$

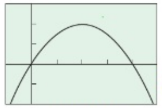
$x=\frac{5}{\sqrt{2}}$

$x^2+y^2=25$

$\frac{25}{2}+y^2=\frac{25}{1}-\frac{25}{2}$

$y^2=\frac{50}{2}-\frac{25}{2}=\frac{25}{2}$

$y=\frac{5}{\sqrt{2}}$



$[-5, 25]$ by $[-100, 150]$

Figure 4.35 The graph of $f(x) = x(20 - x)$ with domain $(-\infty, \infty)$ has an absolute maximum of 100 at $x = 10$. (Example 1)

$x + y = 20$

$y = 20 - x$

$xy = \text{Max}$

EXAMPLE 1 Using the Strategy

Find two numbers whose sum is 20 and whose product is as large as possible.

SOLUTION

Model If one number is x, the other is $(20 - x)$, and their product is $f(x) = x(20 - x)$.

Solve Graphically We can see from the graph of f in Figure 4.35 that there is a maximum. From what we know about parabolas, the maximum occurs at $x = 10$.

Interpret The two numbers we seek are $x = 10$ and $20 - x = 10$.

Now try Exercise 1.

$20x - x^2 = \text{max}$

$20 - 2x = 0$

$+2x \quad +2x$

$20 = 2x$

$x = 10$

In Exercises 1–10, solve the problem analytically. Support your answer graphically.

$x + y = 20 \rightarrow y = 20 - x$

1. ***Finding Numbers*** The sum of two nonnegative numbers is 20. Find the numbers if

   (a) the sum of their squares is as large as possible; as small as possible.

   (b) one number plus the square root of the other is as large as possible; as small as possible.

a) $x^2 + y^2 = \text{max (or min)}$

$x^2 + (20-x)^2 = \text{max (or min)}$

$x^2 + 400 - 40x + x^2 = \text{max (or min)}$

$2x^2 - 40x + 400 = \text{max (or min)}$

$4x - 40 = 0$

$x = 10 \; ; \; y = 10$

$x = 0 \; ; \; y = 20$

Think: most extreme situation...

$10^2 + 10^2 =$

$100 + 100 = 200$

$0^2 + 20^2$

$0 + 400 = 400$

b) $x + \sqrt{y} = \text{max (or min)}$

$x + \sqrt{20 - x} = \text{max (or min)}$

$x + (20-x)^{\frac{1}{2}} = \text{max (or min)}$

$1 + \frac{1}{2}(20-x)^{-\frac{1}{2}}(-1) = 0$

$x \neq 20$

$1 - \frac{1}{2(20-x)^{\frac{1}{2}}} = 0$

$\frac{2(20-x)^{\frac{1}{2}}}{2(20-x)^{\frac{1}{2}}} - \frac{1}{2(20-x)^{\frac{1}{2}}} = \frac{2(20-x)^{\frac{1}{2}} - 1}{2(20-x)^{\frac{1}{2}}} = 0$

$2(20-x)^{\frac{1}{2}} - 1 = 0$

$\left((20-x)^{\frac{1}{2}}\right)^2 = \left(\frac{1}{2}\right)^2$

$\frac{1}{4} + \frac{3}{4} = \frac{3}{4}$

$20 - x = \frac{1}{4}$

$-x = \frac{1}{4} - \frac{80}{4}$

$x = \frac{79}{4}$

2. ***Maximizing Area*** What is the largest possible area for a right triangle whose hypotenuse is 5 cm long, and what are its dimensions?

3. ***Maximizing Perimeter*** What is the smallest perimeter possible for a rectangle whose area is 16 in^2, and what are its dimensions? Smallest perimeter = 16 in., dimensions are 4 in. by 4 in.

4. ***Finding Area*** Show that among all rectangles with an 8-m perimeter, the one with largest area is a square. See page 232.

③ $2x + 2y = \text{smallest}$

$xy = 16$

$y = \frac{16}{x}$

$2x + 2\left(\frac{16}{x}\right) = \text{smallest perimeter}$

$2x + \frac{32}{x} = \#$ $\quad 32x^{-1}$

$2 - 32x^{-2} = 0 \qquad x = 4, -4$

$2x^2 - 32 = 0$

$x^2 - 16 = 0$

$(x+4)(x-4) = 0$

$P = 8\text{m}$

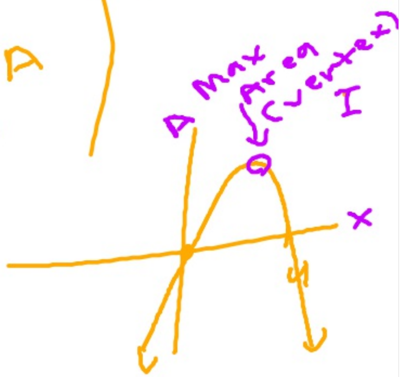
$2x + 2y = 8$

$x + y = 4$

$xy = \text{max} = A$

$A = x(4-x)$

$y = 4 - x$



EXAMPLE 2 Inscribing Rectangles

A rectangle is to be inscribed under one arch of the sine curve (Figure 4.36). What is the largest area the rectangle can have, and what dimensions give that area?

SOLUTION

Model Let $(x, \sin x)$ be the coordinates of point P in Figure 4.36. From what we know about the sine function the x-coordinate of point Q is $(\pi - x)$. Thus,

$$\pi - 2x = \text{length of rectangle}$$

and

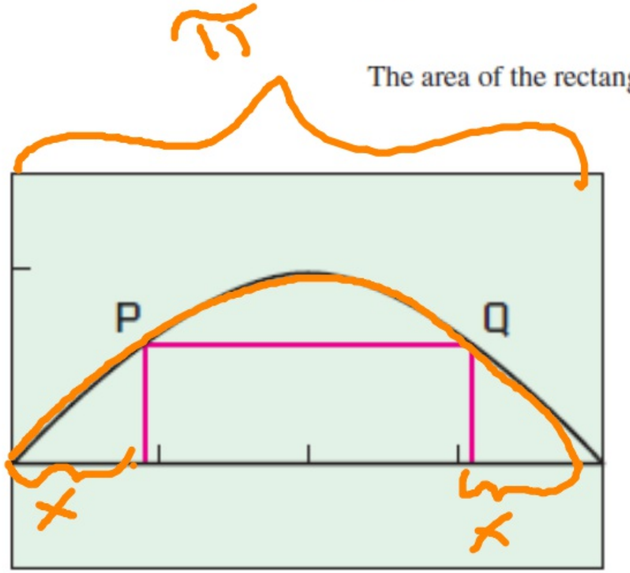
$$\sin x = \text{height of rectangle.}$$

The area of the rectangle is

$$A(x) = (\pi - 2x)\sin x.$$

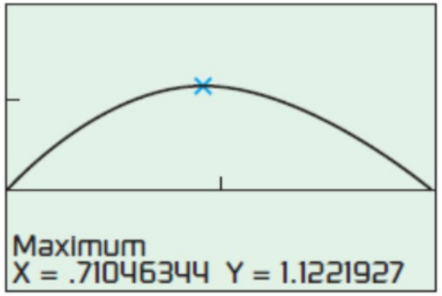
$xy = \text{Area}$

$(\sin x)$



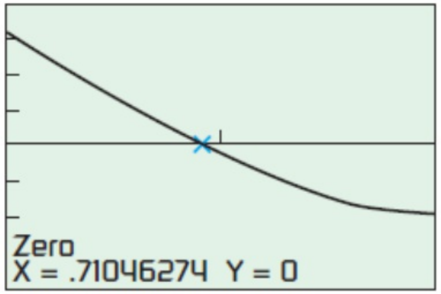
[0, π] by [–0.5, 1.5]

Figure 4.36 A rectangle inscribed under one arch of $y = \sin x$. (Example 2)



[0, π/2] by [–1, 2]

(a)



[0, π/2] by [–4, 4]

(b)

Solve Analytically and Graphically We can assume that $0 \le x \le \pi/2$. Notice that $A = 0$ at the endpoints $x = 0$ and $x = \pi/2$. Since A is differentiable, the only critical points occur at the zeros of the first derivative,

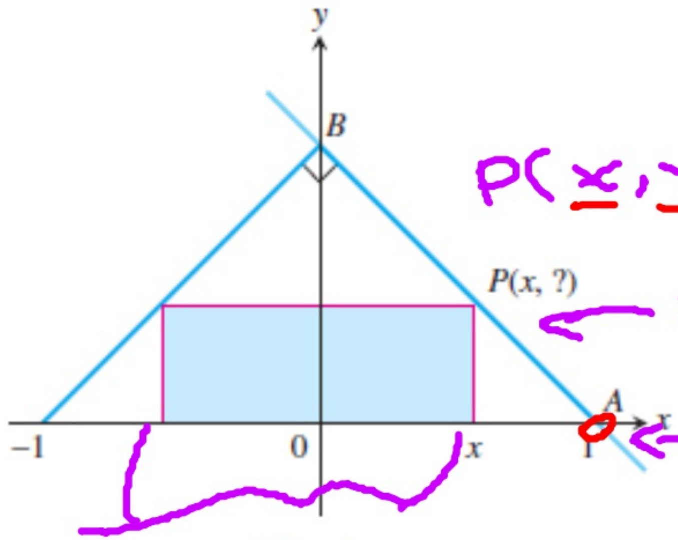
$$A'(x) = -2 \sin x + (\pi - 2x) \cos x.$$

It is not possible to solve the equation $A'(x) = 0$ using algebraic methods. We can use the graph of A (Figure 4.37a) to find the maximum value and where it occurs. Or, we can use the graph of A' (Figure 4.37b) to find where the derivative is zero, and then evaluate A at this value of x to find the maximum value. The two x-values appear to be the same, as they should.

Interpret The rectangle has a maximum area of about 1.12 square units when $x \approx 0.71$. At this point, the rectangle is $\pi - 2x \approx 1.72$ units long by $\sin x \approx 0.65$ unit high.

Now try Exercise 5.

5. *Inscribing Rectangles* The figure shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.



$2x$

$y = mx + b$

$0 = -1(1) + b$

$0 = -1 + b$

$b = 1$

$y = -x + 1$

$P(x, y)$

$m = -1$

$(1, 0)$

$A = bh$

b) $b = 2x$

$h = (1 - x)$

$A = (2x)(1 - x)$

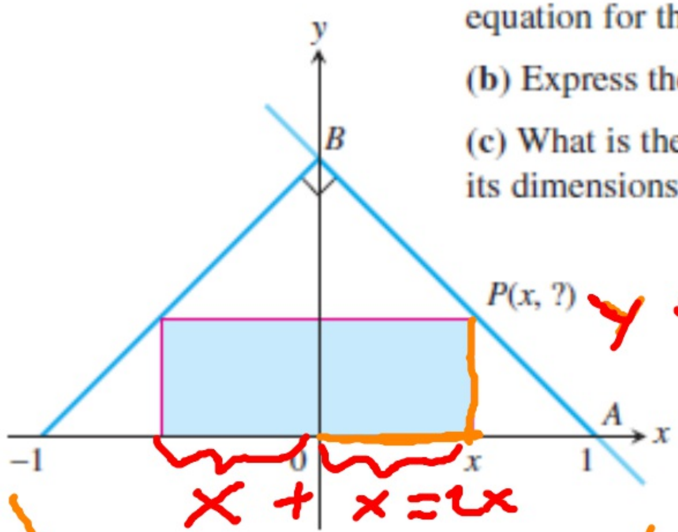
(a) Express the y-coordinate of P in terms of x. [*Hint:* Write an equation for the line AB.] $y = 1 - x$

(b) Express the area of the rectangle in terms of x. $A(x) = 2x(1 - x)$

(c) What is the largest area the rectangle can have, and what are its dimensions? Largest area $= \frac{1}{2}$, dimensions are 1 by $\frac{1}{2}$

c)

5. *Inscribing Rectangles* The figure shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.



$x + x = 2x$

2

$\sqrt{2}$

(a) Express the y-coordinate of P in terms of x. [*Hint:* Write an equation for the line AB.] $y = 1 - x$

(b) Express the area of the rectangle in terms of x. $A(x) = 2x(1 - x)$

(c) What is the largest area the rectangle can have, and what are its dimensions? Largest area $= \frac{1}{2}$, dimensions are 1 by $\frac{1}{2}$

$y = 1 - x$

length x width

(largest) Area =

$A(x) = (2x)(1 - x)$

(largest) $A(x) = 2x - 2x^2$

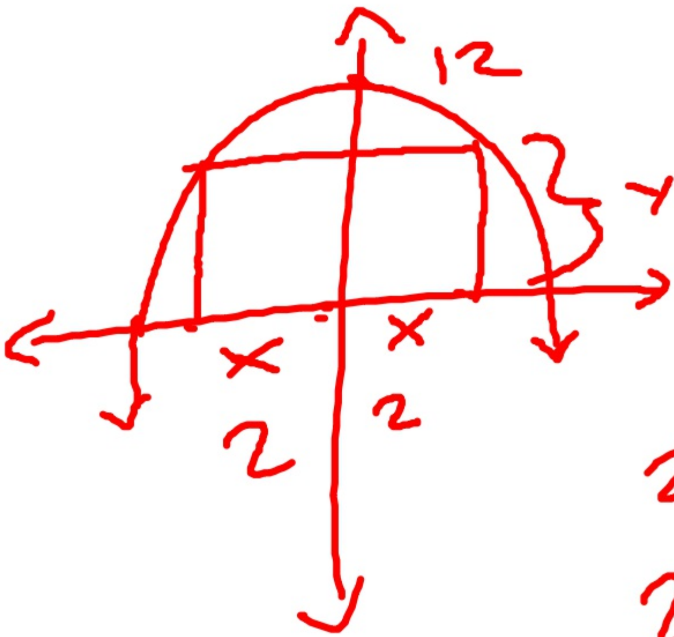
$0 = 2 - 4x$

$\frac{4x}{4} = \frac{2}{4}$

$x = \frac{1}{2}$

6. *Largest Rectangle* A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions? Largest area = 32, dimensions are 4 by 8

$12 - 4 = 8$



$y = 12 - x^2$

$(2x)(12 - x^2) = \#$

$24x - 2x^3 = \#$

$24 - 6x^2 = 0$

$x^2 = 4$

$x = 2$

EXAMPLE 3 Fabricating a Box

An open-top box is to be made by cutting congruent squares of side length x from the corners of a 20- by 25-inch sheet of tin and bending up the sides (Figure 4.38). How large should the squares be to make the box hold as much as possible? What is the resulting maximum volume?

SOLUTION

Model The height of the box is x, and the other two dimensions are $(20 - 2x)$ and $(25 - 2x)$. Thus, the volume of the box is

$$V(x) = x(20 - 2x)(25 - 2x).$$

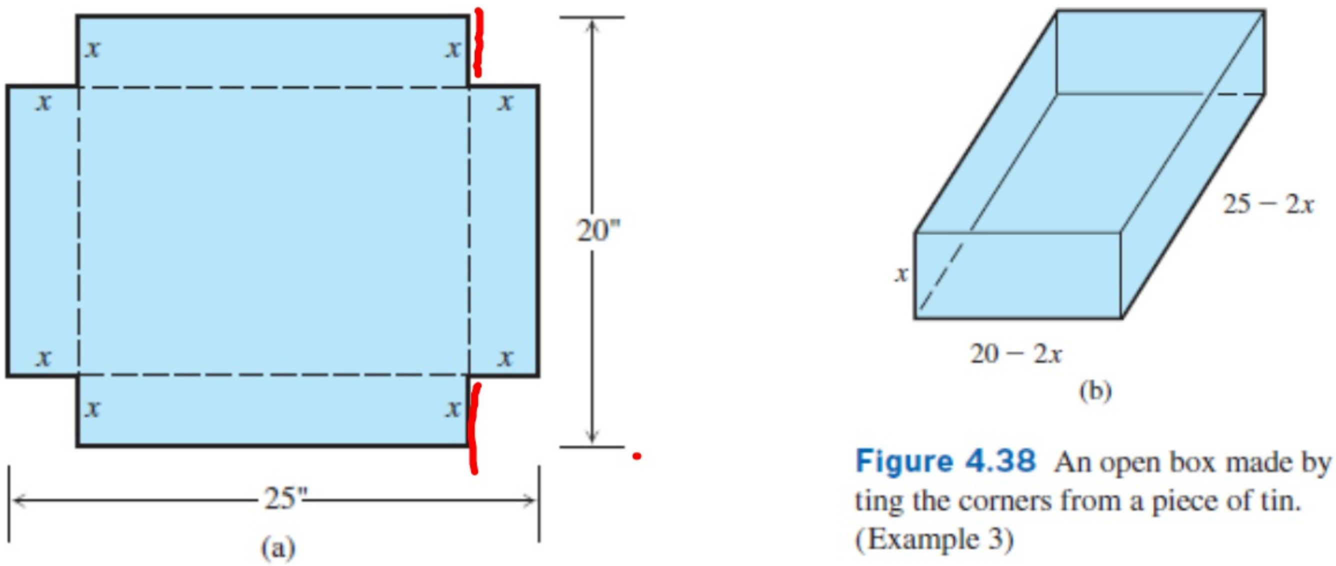


Figure 4.38 An open box made by cutting the corners from a piece of tin. (Example 3)

Solve Graphically Because $2x$ cannot exceed 20, we have $0 \leq x \leq 10$. Figure 4.39 suggests that the maximum value of V is about 820.53 and occurs at $x \approx 3.68$.

Confirm Analytically Expanding, we obtain $V(x) = 4x^3 - 90x^2 + 500x$. The first derivative of V is

$$V'(x) = 12x^2 - 180x + 500.$$

The two solutions of the quadratic equation $V'(x) = 0$ are

$$c_1 = \frac{180 - \sqrt{180^2 - 48(500)}}{24} \approx 3.68 \qquad \text{and}$$

$$c_2 = \frac{180 + \sqrt{180^2 - 48(500)}}{24} \approx 11.32.$$

Only c_1 is in the domain $[0, 10]$ of V. The values of V at this one critical point and the two endpoints are

Critical point value: $V(c_1) \approx 820.53$

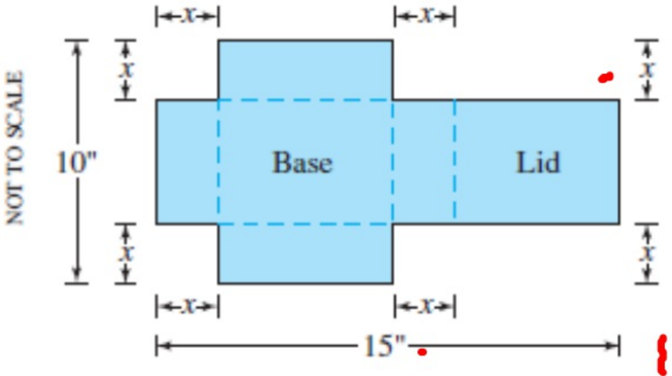
Endpoint values: $V(0) = 0, \quad V(10) = 0.$

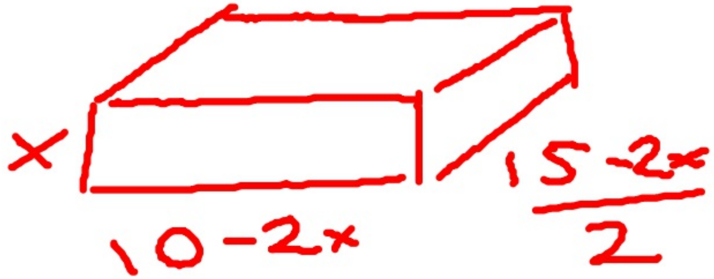
Interpret Cutout squares that are about 3.68 in. on a side give the maximum volume, about 820.53 in^3. ***Now try Exercise 7.***

18. ***Designing a Box with Lid*** A piece of cardboard measures 10- by 15-in. Two equal squares are removed from the corners of a 10-in. side as shown in the figure. Two equal rectangles are removed from the other corners so that the tabs can be folded to form a rectangular box with lid.

Domain $(0, 5)$

$10-2x$





$\dfrac{15-2x}{2}$

(a) Write a formula $V(x)$ for the volume of the box.

(b) Find the domain of V for the problem situation and graph V over this domain.

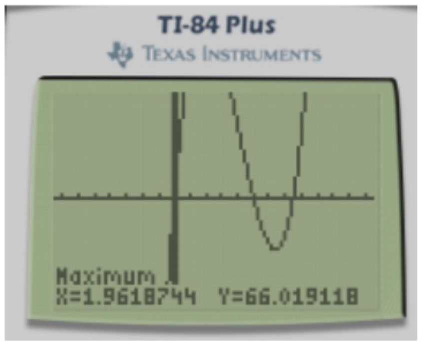
(c) Use a graphical method to find the maximum volume and the value of x that gives it.

(d) Confirm your result in part (c) analytically.

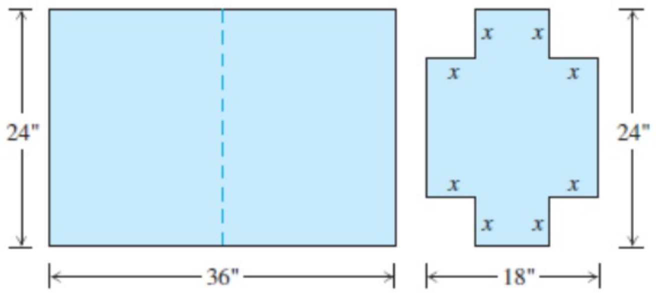
$V = \left(\dfrac{15-2x}{2}\right)(10-2x)(x)$

"confirm analytically"-

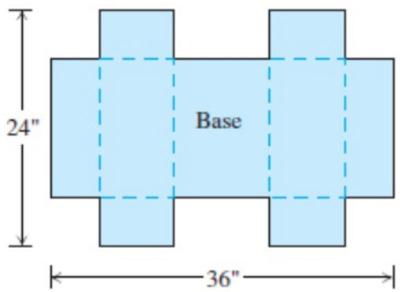
derive V(x), set equal to zero,
find x withinthe interval (0,5)



19. ***Designing a Suitcase*** A 24- by 36-in. sheet of cardboard is folded in half to form a 24- by 18-in. rectangle as shown in the figure. Then four congruent squares of side length x are cut from the corners of the folded rectangle. The sheet is unfolded, and the six tabs are folded up to form a box with sides and a lid.



The sheet is then unfolded.



(a) Write a formula $V(x)$ for the volume of the box.

(b) Find the domain of V for the problem situation and graph V over this domain.

(c) Use a graphical method to find the maximum volume and the value of x that gives it.

(d) Confirm your result in part (c) analytically.

(e) Find a value of x that yields a volume of 1120 in^3.

(f) **Writing to Learn** Write a paragraph describing the issues that arise in part (b).

7. ***Optimal Dimensions*** You are planning to make an open rectangular box from an 8- by 15-in. piece of cardboard by cutting congruent squares from the corners and folding up the sides. What are the dimensions of the box of largest volume you can make this way, and what is its volume? See page 232.

8. ***Closing Off the First Quadrant*** You are planning to close off a corner of the first quadrant with a line segment 20 units long running from $(a, 0)$ to $(0, b)$. Show that the area of the triangle enclosed by the segment is largest when $a = b$. See page 232.

9. ***The Best Fencing Plan*** A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions? Largest area = 80,000 m^2; dimensions: 200 m (perpendicular to river) by 400 m (parallel to river)

10. ***The Shortest Fence*** A 216-m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed? Dimensions: 12 m (divider is this length) by 18 m; total length required: 72 m

13. ***Designing a Poster*** You are designing a rectangular poster to contain 50 in^2 of printing with a 4-in. margin at the top and bottom and a 2-in. margin at each side. What overall dimensions will minimize the amount of paper used? 18 in. high by 9 in. wide

14. ***Vertical Motion*** The height of an object moving vertically is given by

$$s = -16t^2 + 96t + 112,$$

(a) 96 ft/sec
(b) 256 feet at $t = 3$ seconds
(c) -128 ft/sec

with s in ft and t in sec. Find **(a)** the object's velocity when $t = 0$, **(b)** its maximum height and when it occurs, and **(c)** its velocity when $s = 0$.

15. ***Finding an Angle*** Two sides of a triangle have lengths a and b, and the angle between them is θ. What value of θ will maximize the triangle's area? [*Hint:* $A = (1/2)\, ab \sin \theta$.] $\theta = \dfrac{\pi}{2}$

13. *Designing a Poster* You are designing a rectangular poster to contain 50 in^2 of printing with a 4-in. margin at the top and bottom and a 2-in. margin at each side. What overall dimensions will minimize the amount of paper used? 18 in. high by 9 in. wide

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(a) 96 ft/sec
(b) 256 feet at $t = 3$ seconds
(c) -128 ft/sec

with s in ft and t in sec. Find **(a)** the object's velocity when $t = 0$, **(b)** its maximum height and when it occurs, and **(c)** its velocity when $s = 0$.

15. *Finding an Angle* Two sides of a triangle have lengths a and b, and the angle between them is θ. What value of θ will maximize the triangle's area? [*Hint:* $A = (1/2)\,ab \sin \theta$.] $\theta = \frac{\pi}{2}$

16. *Designing a Can* What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of 1000 cm^3? Compare the result here with the result in Example 4. Radius = height = $10\pi^{-1/3}$ cm $\approx$ 6.83 cm. In Example 4, because of the top on the can, the "best" design is less big around and taller.

16. ***Designing a Can*** What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of 1000 cm^3? Compare the result here with the result in Example 4. Radius = height = $10\pi^{-1/3}$ cm $\approx$ 6.83 cm. In Example 4, because of the top on the can, the "best" design is less big around and taller.

17. ***Designing a Can*** You are designing a 1000-cm^3 right circular cylindrical can whose manufacture will take waste into account. There is no waste in cutting the aluminum for the side, but the top and bottom of radius r will be cut from squares that measure $2r$ units on a side. The total amount of aluminum used up by the can will therefore be

$$A = 8r^2 + 2\pi rh$$

rather than the $A = 2\pi r^2 + 2\pi rh$ in Example 4. In Example 4 the ratio of h to r for the most economical can was 2 to 1. What is the ratio now? $\frac{8}{\pi}$ to 1