5-5 Solving Polynomial Equations

KeyConcept Sum and Difference of Cubes

Factoring Technique	General Case
Sum of Two Cubes	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
Difference of Two Cubes	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Polynomials that cannot be factored are called **prime polynomials**.

Example 1 Sum and Difference of Cubes

Factor each polynomial. If the polynomial cannot be factored, write *prime*.

a. $16x^4 + 54xy^3$

$16x^4 + 54xy^3 = 2x(8x^3 + 27y^3)$ — Factor out the GCF.

$8x^3$ and $27y^3$ are both perfect cubes, so we can factor the sum of two cubes.

$$\begin{aligned} 8x^3 + 27y^3 &= (2x)^3 + (3y)^3 && (2x)^3 = 8x^3;\ (3y)^3 = 27y^3 \\ &= (2x + 3y)[(2x)^2 - (2x)(3y) + (3y)^2] && \text{Sum of two cubes} \\ &= (2x + 3y)(4x^2 - 6xy + 9y^2) && \text{Simplify.} \end{aligned}$$

$16x^4 + 54xy^3 = 2x(2x + 3y)(4x^2 - 6xy + 9y^2)$ — Replace the GCF.

b. $8y^3 + 5x^3$

The first term is a perfect cube, but the second term is not. So, the polynomial cannot be factored using the sum of two cubes pattern. The polynomial also cannot be factored using quadratic methods or the GCF. Therefore, it is a prime polynomial.

Example 2 Factoring by Grouping

Factor each polynomial. If the polynomial cannot be factored, write *prime*.

a. $8ax + 4bx + 4cx + 6ay + 3by + 3cy$

b. $20fy - 16fz + 15gy + 8hz - 10hy - 12gz$

Example 3 Combine Cubes and Squares

Factor each polynomial. If the polynomial cannot be factored, write *prime*.

a. $x^6 - y^6$

b. $a^3x^2 - 6a^3x + 9a^3 - b^3x^2 + 6b^3x - 9b^3$

Examples 1–3 **Factor completely. If the polynomial is not factorable, write *prime*.**

1. $3ax + 2ay - az + 3bx + 2by - bz$ $(a + b)(3x + 2y - z)$

2. $2kx + 4mx - 2nx - 3ky - 6my + 3ny$ $(2x - 3y) \cdot (k + 2m - n)$

3. $2x^3 + 5y^3$ prime

4. $16g^3 + 2h^3$ $2(2g + h)(4g^2 - 2gh + h^2)$

5. $12qw^3 - 12q^4$ $12q(w - q)(w^2 + qw + q^2)$

6. $3w^2 + 5x^2 - 6y^2 + 2z^2 + 7a^2 - 9b^2$ prime

7. $a^6x^2 - b^6x^2$ $x^2(a - b)(a^2 + ab + b^2) \cdot (a + b)(a^2 - ab + b^2)$

8. $x^3y^2 - 8x^3y + 16x^3 + y^5 - 8y^4 + 16y^3$

9. $8c^3 - 125d^3$ $(2c - 5d)(4c^2 + 10cd + 25d^2)$

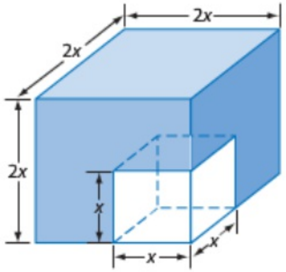
10. $6bx + 12cx + 18dx - by - 2cy - 3dy$ $(6x - y)(b + 2c + 3d)$

8. $(x + y)(x^2 - xy + y^2) \cdot (y - 4)^2$

Example 4 **Solve each equation.**

Real-World Example 4 Solve Polynomial Functions by Factoring

GEOMETRY **Refer to the beginning of the lesson. If the small cube is half the length of the larger cube and the figure is 7000 cubic centimeters, what should be the dimensions of the cubes?**



Since the length of the smaller cube is half the length of the larger cube, then their lengths can be represented by x and $2x$, respectively. The volume of the object equals the volume of the larger cube minus the volume of the smaller cube.

$(2x)^3 - x^3 = 7000$	Volume of object
$8x^3 - x^3 = 7000$	$(2x)^3 = 8x^3$
$7x^3 = 7000$	Subtract.
$x^3 = 1000$	Divide.
$x^3 - 1000 = 0$	Subtract 1000 from each side.
$(x - 10)(x^2 + 10x + 100) = 0$	Difference of cubes
$x - 10 = 0$ or $x^2 + 10x + 100 = 0$	Zero Product Property
$x = 10$ $x = -5 \pm 5i\sqrt{3}$	

Since 10 is the only real solution, the lengths of the cubes are 10 cm and 20 cm.

$(y - 4)^2$

Example 4 **Solve each equation.**

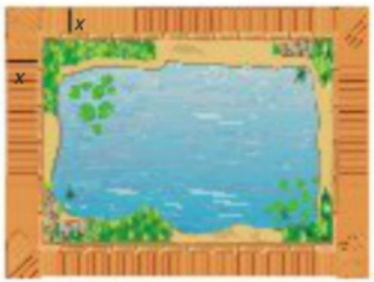
11. $x^4 - 19x^2 + 48 = 0$ **$4, -4, \pm\sqrt{3}$**

12. $x^3 - 64 = 0$ **$4, -2 \pm 2i\sqrt{3}$**

13. $x^3 + 27 = 0$ **$-3, \frac{3 \pm 3i\sqrt{3}}{2}$**

14. $x^4 - 33x^2 + 200 = 0$ **$5, -5, \pm 2\sqrt{2}$**

15. **CCSS PERSEVERANCE** A boardwalk that is x feet wide is built around a rectangular pond. The pond is 30 feet wide and 40 feet long. The combined area of the pond and the boardwalk is 2000 square feet. What is the width of the boardwalk? **5 ft**



Example 5 **Write each expression in quadratic form, if possible.**

Example 5 Quadratic Form

Write each expression in quadratic form, if possible.

a. $\mathbf{150n^8 + 40n^4 - 15}$

$150n^8 + 40n^4 - 15 = 6(5n^4)^2 + 8(5n^4) - 15$ $\quad (5n^4)^2 = 25n^8$

b. $\mathbf{y^8 + 12y^3 + 8}$

This cannot be written in quadratic form since $y^8 \neq (y^3)^2$.

Example 5 **Write each expression in quadratic form, if possible.**

16. $4x^6 - 2x^3 + 8$ $\mathbf{(2x^3)^2 - 1(2x^3) + 8}$

17. $25y^6 - 5y^2 + 20$ **not possible**

Example 6 Solve Equations in Quadratic Form

Solve $18x^4 - 21x^2 + 3 = 0$.

$18x^4 - 21x^2 + 3 = 0$	Original equation
$2(3x^2)^2 - 7(3x^2) + 3 = 0$	$2(3x^2)^2 = 18x^4$
$2u^2 - 7u + 3 = 0$	Let $u = 3x^2$.
$(2u - 1)(u - 3) = 0$	Factor.
$u = \frac{1}{2}$ or $u = 3$	Zero Product Property
$3x^2 = \frac{1}{2}$ $\quad$ $3x^2 = 3$	Replace u with $3x^2$.
$x^2 = \frac{1}{6}$ $\quad$ $x^2 = 1$	Divide by 3.
$x = \pm\frac{\sqrt{6}}{6}$ $\quad$ $x = \pm 1$	Take the square root.

The solutions of the equation are $1, -1, \frac{\sqrt{6}}{6}$, and $-\frac{\sqrt{6}}{6}$.

Example 6 **Solve each equation.**

18. $x^4 - 6x^2 + 8 = 0$ **$2, -2, \sqrt{2}, -\sqrt{2}$**

19. $y^4 - 18y^2 + 72 = 0$ **$\sqrt{6}, -\sqrt{6}, 2\sqrt{3}, -2\sqrt{3}$**

Practice and Problem Solving

Extra Practice is on page R5.

Examples 1–3 Factor completely. If the polynomial is not factorable, write *prime*.

20. $8c^3 - 27d^3$ **21.** $64x^4 + xy^3$ **22.** $a^8 - a^2b^6$

23. $x^6y^3 + y^9$ **24.** $18x^6 + 5y^6$ **prime** **25.** $w^3 - 2y^3$ **prime**

26. $gx^2 - 3hx^2 - 6fy^2 - gy^2 + 6fx^2 + 3hy^2$ **$(x + y)(x - y)(6f + g - 3h)$**

27. $12ax^2 - 20cy^2 - 18bx^2 - 10ay^2 + 15by^2 + 24cx^2$ **$(6x^2 - 5y^2)(2a - 3b + 4c)$**

28. $a^3x^2 - 16a^3x + 64a^3 - b^3x^2 + 16b^3x - 64b^3$ **$(a - b)(a^2 + ab + b^2)(x - 8)^2$**

29. $8x^5 - 25y^3 + 80x^4 - x^2y^3 + 200x^3 - 10xy^3$ **$(2x - y)(4x^2 + 2xy + y^2)(x + 5)^2$**

20. $(2c - 3d) \cdot (4c^2 + 6cd + 9d^2)$

21. $x(4x + y) \cdot (16x^2 - 4xy + y^2)$

22. $a^2(a - b)(a^2 + ab + b^2)(a + b) \cdot (a^2 - ab + b^2)$

23. $y^3(x^2 + y^2) \cdot (x^4 - x^2y^2 + y^4)$

Example 4 Solve each equation. **30.** $3, -3, \pm i\sqrt{10}$ **31.** $6, -6, \pm 2i\sqrt{5}$ **32.** $\pm\sqrt{11}, \pm 2i$

30. $x^4 + x^2 - 90 = 0$ **31.** $x^4 - 16x^2 - 720 = 0$ **32.** $x^4 - 7x^2 - 44 = 0$

33. $x^4 + 6x^2 - 91 = 0$ **34.** $x^3 + 216 = 0$ **35.** $64x^3 + 1 = 0$ $-\frac{1}{4}, \frac{1 \pm i\sqrt{3}}{8}$

33. $\pm\sqrt{7}, \pm i\sqrt{13}$

34. $-6, 3 \pm 3i\sqrt{3}$

Example 5 Write each expression in quadratic form, if possible. **36–41. See margin.**

36. $x^4 + 12x^2 - 8$ **37.** $-15x^4 + 18x^2 - 4$ **38.** $8x^6 + 6x^3 + 7$

39. $5x^6 - 2x^2 + 8$ **40.** $9x^8 - 21x^4 + 12$ **41.** $16x^{10} + 2x^5 + 6$

Example 6 Solve each equation. **42–47. See margin.**

42. $x^4 + 6x^2 + 5 = 0$ **43.** $x^4 - 3x^2 - 10 = 0$ **44.** $4x^4 - 14x^2 + 12 = 0$

45. $9x^4 - 27x^2 + 20 = 0$ **46.** $4x^4 - 5x^2 - 6 = 0$ **47.** $24x^4 + 14x^2 - 3 = 0$

Additional Answers

36. $(x^2)^2 + 12(x^2) - 8$

37. $-15(x^2)^2 + 18(x^2) - 4$

38. $2(2x^3)^2 + 3(2x^3) + 7$

39. not possible

40. $(3x^4)^2 - 7(3x^4) + 12$

41. $4(2x^5)^2 + 1(2x^5) + 6$

42. $\pm i\sqrt{5}, \pm i$

43. $\pm\sqrt{5}, \pm i\sqrt{2}$

44. $\pm\sqrt{2}, \pm\frac{\sqrt{6}}{2}$

45. $\pm\frac{2\sqrt{3}}{3}, \pm\frac{\sqrt{15}}{3}$

46. $\pm\sqrt{2}, \pm i\frac{\sqrt{3}}{2}$

47. $\pm\frac{\sqrt{6}}{6}, \pm i\frac{\sqrt{3}}{2}$