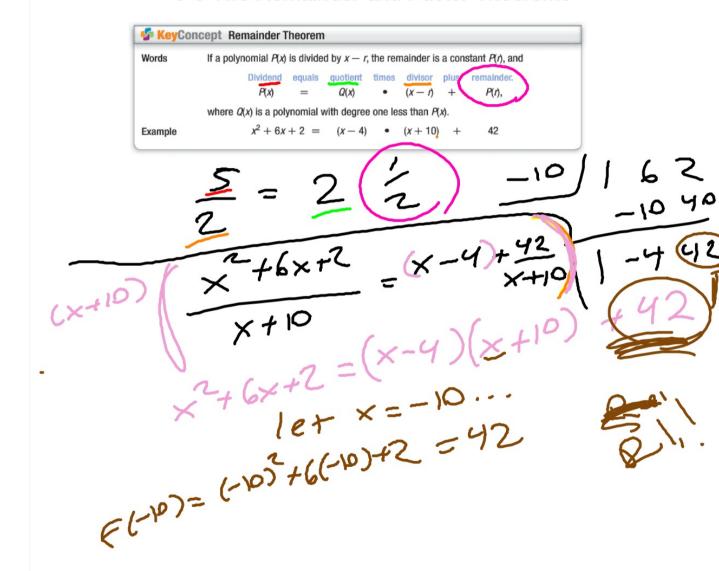
# 5-6 The Remainder and Factor Theorems



# Method 1 Long Division

$$\begin{array}{r}
-3x - 4 \\
x - 3) - 3x^2 + 5x + 4 \\
\underline{-3x^2 + 9x} \\
-4x + 4 \\
\underline{-4x + 12} \\
-8
\end{array}$$

Compare the remainder of -8 to f(3).

$$f(3) = -3(3)^2 + 5(3) + 4$$
 Replace x with 3.  
= -27 + 15 + 4 Multiply.  
= -8 Simplify.

Method 2 Synthetic Division

Notice that the value of f(3) is the same as the remainder when the polynomial is divided by x - 3. This illustrates the **Remainder Theorem**.

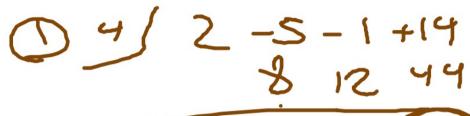
### **Example 1** Synthetic Substitution

If  $f(x) = 3x^4 - 2x^3 + 5x + 2$ , find f(4).

## Method 1 Synthetic Substitution

By the Remainder Theorem, f(4) should be the remainder when the polynomial is divided by x - 4.

Because there is no  $x^2$  term, a zero is placed in this position as a placeholder.



## Method 2 Direct Substitution

Replace x with 4.

$$f(x) = 3x^4 - 2x^3 + 5x + 2$$

Original function

$$f(4) = 3(4)^4 - 2(4)^3 + 5(4) + 2$$

Replace x with 4.

$$= 768 - 128 + 20 + 2 \text{ or } 662$$
 Simplify.

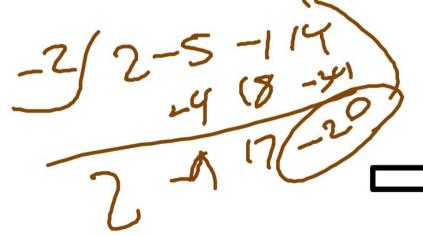
The remainder is 662. Therefore, by using synthetic substitution, f(4) = 662.

By using direct substitution, f(4) = 662. Both methods give the same result

#### Use synthetic substitution to find f(4) and f(-2) for each function. Example 1

**1.** 
$$f(x) = 2x^3 - 5x^2 - x + 14$$
 **58**; **-20**

**2.** 
$$f(x) = x^4 + 8x^3 + x^2 - 4x - 10$$



The number of college students from the United States who study abroad can be modeled by the function S(x) = 0.02x<sup>4</sup> - 0.52x<sup>3</sup> + 4.03x<sup>2</sup> + 0.09x + 77.54, where x is the number of years since 1993 and S(x) is the number of students in thousands.

# Real-World Example 2 Find Function Values



**COLLEGE** Refer to the beginning of the lesson. How many U.S. college students will study abroad in 2018?

Use synthetic substitution to divide  $0.02x^4 - 0.52x^3 + 4.03x^2 + 0.09x + 77.54$  by x - 20.

In 2018, there will be about 2,286,040 U.S. college students studying abroad.

**3. NATURE** The approximate number of bald eagle nesting pairs in the United States can be modeled by the function  $P(x) = -0.16x^3 + 15.83x^2 - 154.15x + 1147.97$ , where x is the number of years since 1970. About how many nesting pairs of bald eagles can be expected in 2018? **12,526** 

**Factors of Polynomials** The synthetic division below shows that the quotient of  $2x^3 - 3x^2 - 17x + 30$  and x + 3 is  $2x^2 - 9x + 10$ .

When you divide a polynomial by one of its binomial factors, the quotient is called a depressed polynomial. A **depressed polynomial** has a degree that is one less than the original polynomial. From the results of the division, and by using the Remainder Theorem, we can make the following statement.

Dividend equals quotient times divisor plus remainder. 
$$2x^3 - 3x^2 - 17x + 30 = (2x^2 - 9x + 10) \cdot (x + 3) + 0$$

Since the remainder is 0, f(-3) = 0. This means that x + 3 is a factor of  $2x^3 - 3x^2 - 17x + 3$ . This illustrates the **Factor Theorem**, which is a special case of the Remainder Theorem.

# **KeyConcept** Factor Theorem

The binomial x - r is a factor of the polynomial P(x) if and only if P(r) = 0.

The Factor Theorem can be used to determine whether a binomial is a factor of a polynomial. It can also be used to determine all of the factors of a polynomial.

# Example 3 Use the Factor Theorem

Determine whether x - 5 is a factor of  $x^3 - 7x^2 + 7x + 15$ . Then find the remaining factors of the polynomial.

The binomial x - 5 is a factor of the polynomial if 5 is a zero of the related polynomial function. Use the Factor Theorem and synthetic division.

Because the remainder is 0, x - 5 is a factor of the polynomial. The polynomial  $x^3 - 7x^2 + 7x + 15$  can be factored as  $(x - 5)(x^2 - 2x - 3)$ . The polynomial  $x^2 - 2x - 3$  is the depressed polynomial. Check to see if this polynomial can be factored.

$$x^2 - 2x - 3 = (x + 1)(x - 3)$$
 Factor the trinomial.  
So,  $x^3 - 7x^2 + 7x + 15 = (x - 5)(x + 1)(x - 3)$ .

You can check your answer by multiplying out the factors and seeing if you come up with the initial polynomial.



### Given a polynomial and one of its factors, find the remaining factors of the polynomial. Example 3

**4.** 
$$x^3 - 6x^2 + 11x - 6$$
;  $x - 1$ 

**4.** 
$$x^3 - 6x^2 + 11x - 6$$
;  $x - 1$  **5.**  $x^3 + x^2 - 16x - 16$ ;  $x + 1$  **7.**  $2x^3 - 5x^2 - 28x + 15$ ;  $x + 3$  **7.**  $2x - 1$ 

**6.** 
$$3x^3 + 10x^2 - x - 12$$
;  $x - 1$ 

7. 
$$2x^3 - 5x^2 - 28x + 15$$
;  $x + 3$   $x - 5$ ,  $2x -$ 

Use synthetic substitution to find f(-5) and f(2) for each function. Example 1

**8.** 
$$f(x) = x^3 + 2x^2 - 3x + 1$$
 **-59; 11 9.**  $f(x) = x^2 - 8x + 6$  **71; -6**

**9.** 
$$f(x) = x^2 - 8x + 6$$
 **71**; **-6**

**10.** 
$$f(x) = 3x^4 + x^3 - 2x^2 + x + 12$$
 **1707**; **62**

**10.** 
$$f(x) = 3x^4 + x^3 - 2x^2 + x + 12$$
 **1707**; **62 11.**  $f(x) = 2x^3 - 8x^2 - 2x + 5$  **-435**; **-15**

**12.** 
$$f(x) = x^3 - 5x + 2$$
 **-98:** 0

**13.** 
$$f(x) = x^5 + 8x^3 + 2x - 15$$
 **-4150: 85**

**14.** 
$$f(x) = x^6 - 4x^4 + 3x^2 - 10$$
 **13.190; 2 15.**  $f(x) = x^4 - 6x - 8$  **647; -4**

**15.** 
$$f(x) = x^4 - 6x - 8$$
 **647**; **-4**

16. FINANCIAL LITERACY A specific car's fuel economy in miles per gallon can be Example 2 approximated by  $f(x) = 0.00000056x^4 - 0.000018x^3 - 0.016x^2 + 1.38x - 0.38$ , where x represents the car's speed in miles per hour. Determine the fuel economy when the car is traveling 40, 50 and 60 miles per hour. 29.5 mpg; 29.87 mpg; 28.19 mpg

Given a polynomial and one of its factors, find the remaining factors of the Example 3 polynomial. 21. x + 6, 2x + 7 23. x + 1,  $x^2 + 2x + 3$ 

17. 
$$x^3 - 3x + 2$$
;  $x + 2$  (x – 1)<sup>2</sup>

17. 
$$x^3 - 3x + 2$$
;  $x + 2$   $(x - 1)^2$  18.  $x^4 + 2x^3 - 8x - 16$ ;  $x + 2$   $x - 2$ ,  $x^2 + 2x + 4$ 

**19.** 
$$x^3 - x^2 - 10x - 8$$
;  $x + 2$  **x - 4**,  $x + 1$  **20.**  $x^3 - x^2 - 5x - 3$ ;  $x - 3$  (x + 1)<sup>2</sup>

**20.** 
$$x^3 - x^2 - 5x - 3$$
;  $x - 3$   $(x + 1)^2$ 

**21.** 
$$2x^3 + 17x^2 + 23x - 42$$
;  $x - 1$ 

**21.** 
$$2x^3 + 17x^2 + 23x - 42$$
;  $x - 1$  **22.**  $2x^3 + 7x^2 - 53x - 28$ ;  $x - 4$  **22.**  $x + 7$ ,  $x +$