5-6 The Remainder and Factor Theorems

KeyConcept Remainder Theorem

Words If a polynomial $P(x)$ is divided by $x - r$, the remainder is a constant $P(r)$, and

Dividend	equals	quotient	times	divisor	plus	remainder.
$P(x)$	$=$	$Q(x)$	$\cdot$	$(x - r)$	$+$	$P(r)$,

where $Q(x)$ is a polynomial with degree one less than $P(x)$.

Example $x^2 + 6x + 2 = (x - 4) \cdot (x + 10) + 42$

$\frac{5}{2} = 2\frac{1}{2}$

$-10 \,|\, 1 \quad 6 \quad 2$
$\quad\; -10 \quad 40$
$\quad 1 \quad -4 \quad 42$

$(x+10)\left(\frac{x^2+6x+2}{x+10} = (x-4) + \frac{42}{x+10}\right)$

$x^2+6x+2 = (x-4)(x+10) + 42$

let $x = -10 \ldots$

$F(-10) = (-10)^2 + 6(-10) + 2 = 42$

Method 1 **Long Division**

$$
\begin{array}{r}
-3x - 4 \\
x - 3 \overline{)\,-3x^2 + 5x + 4} \\
\underline{-3x^2 + 9x} \\
-4x + 4 \\
\underline{-4x + 12} \\
-8
\end{array}
$$

Method 2 **Synthetic Division**

$$
\begin{array}{c|ccc}
3 & -3 & 5 & 4 \\
 & & -9 & -12 \\
\hline
 & -3 & -4 & \boxed{-8}
\end{array}
$$

$f(3) = -8$

Compare the remainder of -8 to $f(3)$.

$f(3) = -3(3)^2 + 5(3) + 4$ Replace x with 3.

$= -27 + 15 + 4$ Multiply.

$= -8$ Simplify.

Notice that the value of $f(3)$ is the same as the remainder when the polynomial is divided by $x - 3$. This illustrates the **Remainder Theorem**.

Example 1 Synthetic Substitution

If $f(x) = 3x^4 - 2x^3 + 5x + 2$, find $f(4)$.

Method 1 **Synthetic Substitution**

By the Remainder Theorem, $f(4)$ should be the remainder when the polynomial is divided by $x - 4$.

4	3	−2	0	5	2
		12	40	160	660
	3	10	40	165	662

Because there is no x^2 term, a zero is placed in this position as a placeholder.

The remainder is 662. Therefore, by using synthetic substitution, $f(4) = 662$.

Method 2 **Direct Substitution**

Replace x with 4.

$f(x) = 3x^4 - 2x^3 + 5x + 2$ — Original function

$f(4) = 3(4)^4 - 2(4)^3 + 5(4) + 2$ — Replace x with 4.

$= 768 - 128 + 20 + 2$ or 662 — Simplify.

By using direct substitution, $f(4) = 662$. Both methods give the same result.

① 4 | 2 −5 −1 +14
8 12 44
2 3 11 58

Example 1 **Use synthetic substitution to find $f(4)$ and $f(-2)$ for each function.**

1. $f(x) = 2x^3 - 5x^2 - x + 14$ **58; −20**

2. $f(x) = x^4 + 8x^3 + x^2 - 4x - 10$

−2 | 2 −5 −1 14
−4 18 −34
2 −9 17 −20

The number of college students from the United States who study abroad can be modeled by the function $S(x) = 0.02x^4 - 0.52x^3 + 4.03x^2 + 0.09x + 77.54$, where x is the number of years since 1993 and $S(x)$ is the number of students in thousands.

Real-World Example 2 Find Function Values

COLLEGE Refer to the beginning of the lesson. How many U.S. college students will study abroad in 2018?

Use synthetic substitution to divide $0.02x^4 - 0.52x^3 + 4.03x^2 + 0.09x + 77.54$ by $x - 20$.

25	0.02	−0.52	4.03	0.09	77.54
		0.5	−0.5	88.25	2208.5
	0.02	−0.02	3.53	88.34	2286.04

In 2018, there will be about 2,286,040 U.S. college students studying abroad.

Example 2

3. **NATURE** The approximate number of bald eagle nesting pairs in the United States can be modeled by the function $P(x) = -0.16x^3 + 15.83x^2 - 154.15x + 1147.97$, where x is the number of years since 1970. About how many nesting pairs of bald eagles can be expected in 2018? **12,526**

2 Factors of Polynomials The synthetic division below shows that the quotient of $2x^3 - 3x^2 - 17x + 30$ and $x + 3$ is $2x^2 - 9x + 10$.

$$\begin{array}{r|rrrr} -3 & 2 & -3 & -17 & 30 \\ & & -6 & 27 & -30 \\ \hline & 2 & -9 & 10 & \multicolumn{1}{|r}{0} \end{array}$$

When you divide a polynomial by one of its binomial factors, the quotient is called a depressed polynomial. A **depressed polynomial** has a degree that is one less than the original polynomial. From the results of the division, and by using the Remainder Theorem, we can make the following statement.

$$\overbrace{2x^3 - 3x^2 - 17x + 30}^{\text{Dividend}} \quad \overbrace{=}^{\text{equals}} \quad \overbrace{(2x^2 - 9x + 10)}^{\text{quotient}} \quad \overbrace{\cdot}^{\text{times}} \quad \overbrace{(x + 3)}^{\text{divisor}} \quad \overbrace{+}^{\text{plus}} \quad \overbrace{0}^{\text{remainder.}}$$

Since the remainder is 0, $f(-3) = 0$. This means that $x + 3$ is a factor of $2x^3 - 3x^2 - 17x + 3$
This illustrates the **Factor Theorem**, which is a special case of the Remainder Theorem.

KeyConcept Factor Theorem

The binomial $x - r$ is a factor of the polynomial $P(x)$ if and only if $P(r) = 0$.

The Factor Theorem can be used to determine whether a binomial is a factor of a polynomial. It can also be used to determine all of the factors of a polynomial.

Example 3 Use the Factor Theorem

Determine whether $x - 5$ is a factor of $x^3 - 7x^2 + 7x + 15$. Then find the remaining factors of the polynomial.

The binomial $x - 5$ is a factor of the polynomial if 5 is a zero of the related polynomial function. Use the Factor Theorem and synthetic division.

$$\begin{array}{r|rrrr} 5 & 1 & -7 & 7 & 15 \\ & & 5 & -10 & -15 \\ \hline & 1 & -2 & -3 & 0 \end{array}$$

5 -1 | 1 1 -16 -16

-1 0 16

1 0 -16 0

$(x+1)(x^2 - 16)$

Because the remainder is 0, $x - 5$ is a factor of the polynomial. The polynomial $x^3 - 7x^2 + 7x + 15$ can be factored as $(x - 5)(x^2 - 2x - 3)$. The polynomial $x^2 - 2x - 3$ is the depressed polynomial. Check to see if this polynomial can be factored.

$x^2 - 2x - 3 = (x + 1)(x - 3)$ Factor the trinomial.

So, $x^3 - 7x^2 + 7x + 15 = (x - 5)(x + 1)(x - 3)$.

You can check your answer by multiplying out the factors and seeing if you come up with the initial polynomial.

Example 3 **Given a polynomial and one of its factors, find the remaining factors of the polynomial.**

4. $x^3 - 6x^2 + 11x - 6;\ x - 1$

5. $x^3 + x^2 - 16x - 16;\ x + 1$ **$x + 4,\ x - 4$**

6. $3x^3 + 10x^2 - x - 12;\ x - 1$

7. $2x^3 - 5x^2 - 28x + 15;\ x + 3$ **$x - 5,\ 2x - 1$**

Example 1 **Use synthetic substitution to find $f(-5)$ and $f(2)$ for each function.**

8. $f(x) = x^3 + 2x^2 - 3x + 1$ **−59; 11**

9. $f(x) = x^2 - 8x + 6$ **71; −6**

10. $f(x) = 3x^4 + x^3 - 2x^2 + x + 12$ **1707; 62**

11. $f(x) = 2x^3 - 8x^2 - 2x + 5$ **−435; −15**

12. $f(x) = x^3 - 5x + 2$ **−98; 0**

13. $f(x) = x^5 + 8x^3 + 2x - 15$ **−4150; 85**

14. $f(x) = x^6 - 4x^4 + 3x^2 - 10$ **13,190; 2**

15. $f(x) = x^4 - 6x - 8$ **647; −4**

Example 2

16. **FINANCIAL LITERACY** A specific car's fuel economy in miles per gallon can be approximated by $f(x) = 0.00000056x^4 - 0.000018x^3 - 0.016x^2 + 1.38x - 0.38$, where x represents the car's speed in miles per hour. Determine the fuel economy when the car is traveling 40, 50 and 60 miles per hour. **29.5 mpg; 29.87 mpg; 28.19 mpg**

Example 3 **Given a polynomial and one of its factors, find the remaining factors of the polynomial.** **21. $x + 6$, $2x + 7$ 23. $x + 1$, $x^2 + 2x + 3$**

17. $x^3 - 3x + 2$; $x + 2$ **$(x - 1)^2$**

18. $x^4 + 2x^3 - 8x - 16$; $x + 2$ **$x - 2$, $x^2 + 2x + 4$**

19. $x^3 - x^2 - 10x - 8$; $x + 2$ **$x - 4$, $x + 1$**

20. $x^3 - x^2 - 5x - 3$; $x - 3$ **$(x + 1)^2$**

21. $2x^3 + 17x^2 + 23x - 42$; $x - 1$

22. $2x^3 + 7x^2 - 53x - 28$; $x - 4$ **$x + 7$, $2x + 1$**