5-7 Roots and Zeros

ConceptSummary Zeros, Factors, Roots, and Intercepts

Words

Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial function. Then the following statements are equivalent.

- c is a zero of $P(x)$.
- c is a root or solution of $P(x) = 0$.
- $x - c$ is a factor of $a_n x^n + \cdots + a_1 x + a_0$.
- If c is a real number, then $(c, 0)$ is an x-intercept of the graph of $P(x)$.

Example

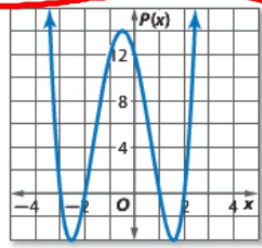
Consider the polynomial function $P(x) = x^4 + 2x^3 - 7x^2 - 8x + 12$.

The zeros of $P(x) = x^4 + 2x^3 - 7x^2 - 8x + 12$ are $-3, -2, 1$, and 2.

The roots of $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$ are $-3, -2, 1$, and 2.

The factors of $x^4 + 2x^3 - 7x^2 - 8x + 12$ are $(x + 3), (x + 2), (x - 1)$, and $(x - 2)$.

The x-intercepts of the graph of $P(x) = x^4 + 2x^3 - 7x^2 - 8x + 12$ are $(-3, 0), (-2, 0), (1, 0)$, and $(2, 0)$.



$\frac{12}{2} = \frac{6}{3}$

$0 = (x+3)(x^3 - x^2 - 4x - 4)$

$0 = (x+2)(x+3)(x^2 - 3x + 2)$

$(x+2)(x+3)(x-1)(x-2)$

$0 = ($

factor completely

When solving a polynomial equation with degree greater than zero, there may be one or more real roots or no real roots (the roots are imaginary numbers). Since real numbers and imaginary numbers both belong to the set of complex numbers, all polynomial equations with degree greater than zero will have at least one root in the set of complex numbers. This is the **Fundamental Theorem of Algebra**.

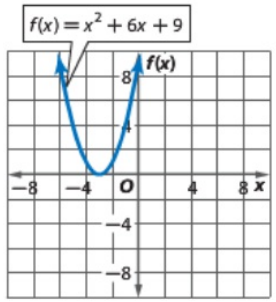
KeyConcept Fundamental Theorem of Algebra

Every polynomial equation with degree greater than zero has at least one root in the set of complex numbers.

Example 1 Determine Number and Type of Roots

Solve each equation. State the number and type of roots.

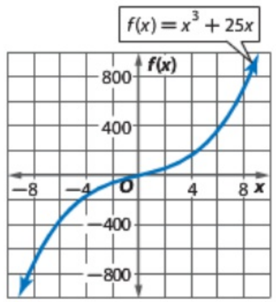
a. $x^2 + 6x + 9 = 0$



ReadingMath

Repeated Roots Polynomial equations can have double roots, triple roots, quadruple roots, and so on. In general, these are referred to as *multiple roots*.

b. $x^3 + 25x = 0$



Example 1 Solve each equation. State the number and type of roots.

1. $x^2 - 3x - 10 = 0$ **$-2, 5$; 2 real**
2. $x^3 + 12x^2 + 32x = 0$ **$-8, -4, 0$; 3 real**
3. $16x^4 - 81 = 0$ **$-\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}i, \frac{3}{2}i$; 2 real, 2 imaginary**
4. $0 = x^3 - 8$ **$2, -1 + i\sqrt{3}, -1 - i\sqrt{3}$; 1 real, 2 imaginary**

(1) $(x-5)(x+2)=0$ $\quad -5, 2$ (-3, -10)

$x - 5 = 0$
$+5 \quad +5$
$x = 5$

$x + 2 = 0$
$-2 \quad -2$
$x = -2$

(3) $16x^4 - 81$

$(4x^2)^2 - (9)^2$

a b

$(4x^2 - 9)(4x^2 + 9) =$

$4x^2 - 9 = 0$
$4x^2 = 9$
$\sqrt{x^2} = \sqrt{\frac{9}{4}}$
$x = \pm 3/2$
2 real

$4x^2 + 9 = 0$
$4x^2 = -9$
$\sqrt{x^2} = \sqrt{\frac{-9}{4}}$
$x = \frac{\pm 3}{2} i$
2 imaginary

KeyConcept Corollary to the Fundamental Theorem of Algebra

Words	A polynomial equation of degree *n* has exactly *n* roots in the set of complex numbers, including repeated roots.
Example	$x^3 + 2x^2 + 6$ — 3 roots; $4x^4 - 3x^3 + 5x - 6$ — 4 roots; $-2x^5 - 3x^2 + 8$ — 5 roots

Similarly, an *n*th degree polynomial function has exactly *n* zeros.

Example 2 Find Numbers of Positive and Negative Zeros

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $f(x) = x^6 + 3x^5 - 4x^4 - 6x^3 + x^2 - 8x + 5$.

Because $f(x)$ has degree 6, it has six zeros, either real or imaginary. Use Descartes' Rule of Signs to determine the possible number and type of *real* zeros.

Count the number of changes in sign for the coefficients of $f(x)$.

$$f(x) = x^6 + 3x^5 - 4x^4 - 6x^3 + x^2 - 8x + 5$$

no (+ to +), yes (+ to −), no (− to −), yes (− to +), yes (+ to −), yes (− to +)

There are 4 sign changes, so there are 4, 2, or 0 positive real zeros.

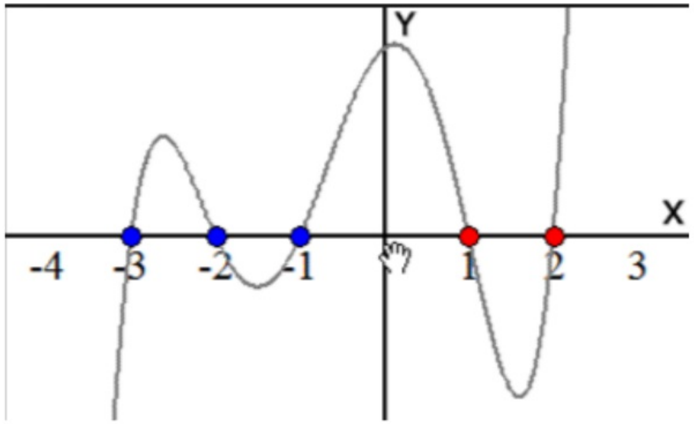
Count the number of changes in sign for the coefficients of $f(-x)$.

$$f(-x) = (-x)^6 + 3(-x)^5 - 4(-x)^4 - 6(-x)^3 + (-x)^2 - 8(-x) + 5$$
$$= x^6 - 3x^5 - 4x^4 + 6x^3 + x^2 + 8x + 5$$

yes (+ to −), no (− to −), yes (− to +), no (+ to +), no (+ to +), no (+ to +)

There are 2 sign changes, so there are 2, or 0 negative real zeros.
Make a chart of the possible combinations of real and imaginary zeros.

Number of Positive Real Zeros	Number of Negative Real Zeros	Number of Imaginary Zeros	Total Number of Zeros
4	2	0	$4 + 2 + 0 = 6$
4	0	2	$4 + 0 + 2 = 6$
2	2	2	$2 + 2 + 2 = 6$
2	0	4	$2 + 0 + 4 = 6$
0	2	4	$0 + 2 + 4 = 6$
0	0	6	$0 + 0 + 6 = 6$



"negative zeros" "positive zeros"

Example 2 State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function. [redacted]

5. $f(x) = x^3 - 2x^2 + 2x - 6$ 3 or 1; 0; 0 or 2

6. $f(x) = 6x^4 + 4x^3 - x^2 - 5x - 7$ [redacted]

7. $f(x) = 3x^5 - 8x^3 + 2x - 4$ 1 or 3; 0 or 2; 0, 2, or 4

8. $f(x) = -2x^4 - 3x^3 - 2x - 5$ [redacted]

(5) $f(-x) = (-x)^3 - 2(-x)^2 + 2(-x)$

-6

$-x^3 - 2x^2 - 2x - 6$

no sign change

neg. 0's

pos. zeros → $f(x) = x^7 \ldots$

7. $f(x) = 3x^5 - 8x^3 + 2x - 4$ **1 or 3; 0 or 2; 0, 2, or 4** 8. $f(x) = -2x^4 - 3x^3 - 2x - 5$ **0; 0 or 2; 2 or 4**

3 or 1 positive zero's

(7) $f(x) = 3x^5 - 8x^3 + 2x - 4$

$f(-x) = -3x^5 + 8x^3 - 2x - 4$

2 or 0 neg. zeros

5 = 3 pos, 2 neg., 0 imaginary

= 1 pos, 2 neg, 2 imaginary

= 3 pos, 0 neg, 2 imaginary

= 1 pos, 0 neg, 4 imaginary

Find all of the zeros of $f(x) = x^4 - 18x^2 + 12x + 80$.

Step 1 Determine the total number of zeros.

Since $f(x)$ has degree 4, the function has 4 zeros.

Step 2 Determine the type of zeros.

Examine the number of sign changes for $f(x)$ and $f(-x)$.

$f(x) = x^4 - 18x^2 + 12x + 80$ — yes, yes, no

$f(-x) = x^4 - 18x^2 - 12x + 80$ — yes, no, yes

Because there are 2 sign changes for the coefficients of $f(x)$, the function has 2 or 0 positive real zeros. Because there are 2 sign changes for the coefficients of $f(-x)$, $f(x)$ has 2 or 0 negative real zeros. Thus, $f(x)$ has 4 real zeros, 2 real zeros and 2 imaginary zeros, or 4 imaginary zeros.

Step 3 Determine the real zeros.

List some possible values, and then use synthetic substitution to evaluate $f(x)$ for real values of x.

x	1	0	−18	12	80
−3	1	−3	−9	39	−37
−2	**1**	**−2**	**−14**	**40**	**0**
−1	1	−1	−17	29	51
0	1	0	−18	12	80
1	1	1	−17	−5	75
2	1	2	−14	−2	76

Each row shows the coefficients of the depressed polynomial and the remainder.

From the table, we can see that one zero occurs at $x = -2$. Since there are 2 negative real zeros, use synthetic substitution with the depressed polynomial function $f(x) = x^3 - 2x^2 - 14x + 40$ to find a second negative zero.

A second negative zero is at $x = -4$. Since the depressed polynomial $x^2 - 6x + 10$ is quadratic, use the Quadratic Formula to find the remaining zeros of $f(x) = x^2 - 6x + 10$.

x	1	−2	−14	40
−4	**1**	**−6**	**10**	**0**
−5	1	−7	21	−65
−6	1	−8	34	−164

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula

$$= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(10)}}{2(1)}$$ Replace a with 1, b with −6, and c with 10.

$$= 3 \pm i$$ Simplify.

The function has zeros at -4, -2, $3 + i$, and $3 - i$.

Example 3 **Find all of the zeros of each function.**

9. $f(x) = x^3 + 9x^2 + 6x - 16$ **−8, −2, 1**

10. $f(x) = x^3 + 7x^2 + 4x + 28$ [redacted]

11. $f(x) = x^4 - 2x^3 - 8x^2 - 32x - 384$ **−4, 6, −4*i*, 4*i***

12. $f(x) = x^4 - 6x^3 + 9x^2 + 6x - 10$ [redacted]

In Chapter 4, you learned that the product of complex conjugates is always a real number and that complex roots always come in conjugate pairs. For example, if one root of $x^2 - 8x + 52 = 0$ is $4 + 6i$, then the other root is $4 - 6i$.

This applies to the zeros of polynomial functions as well. For any polynomial function with real coefficients, if an imaginary number is a zero of that function, its conjugate is also a zero. This is called the **Complex Conjugates Theorem**.

KeyConcept Complex Conjugates Theorem

Words Let a and b be real numbers, and $b \neq 0$. If $a + bi$ is a zero of a polynomial function with real coefficients, then $a - bi$ is also a zero of the function.

Example If $3 + 4i$ is a zero of $f(x) = x^3 - 4x^2 + 13x + 50$, then $3 - 4i$ is also a zero of the function.

When you are given all of the zeros of a polynomial function and are asked to determine the function, convert the zeros to factors and then multiply all of the factors together. The result is the polynomial function.

Example 4 Use Zeros to Write a Polynomial Function

Write a polynomial function of least degree with integral coefficients, the zeros of which include -1 and $5 - i$.

Understand If $5 - i$ is a zero, then $5 + i$ is also a zero according to the Complex Conjugates Theorem. So, $x + 1$, $x - (5 - i)$, and $x - (5 + i)$ are factors of the polynomial.

Plan Write the polynomial function as a product of its factors.

$$P(x) = (x + 1)[x - (5 - i)][x - (5 + i)]$$

Solve Multiply the factors to find the polynomial function.

$P(x) = (x + 1)\,[x - (5 - i)][x - (5 + i)]$	Write the equation.
$= (x + 1)\,[(x - 5) + i][(x - 5) - i]$	Regroup terms.
$= (x + 1)\,[(x - 5)^2 - i^2]$	Difference of squares
$= (x + 1)\,[(x^2 - 10x + 25 - (-1)]$	Square terms.
$= (x + 1)\,(x^2 - 10x + 26)$	Simplify.
$= x^3 - 10x^2 + 26x + x^2 - 10x + 26$	Multiply.
$= x^3 - 9x^2 + 16x + 26$	Combine like terms.

Check Because there are 3 zeros, the degree of the polynomial function must be 3, so $P(x) = x^3 - 9x^2 + 16x + 26$ is a polynomial function of least degree with integral coefficients and zeros of -1, $5 - i$, and $5 + i$.

Example 4 Write a polynomial function of least degree with integral coefficients that have the given zeros.

13. 4, −1, 6 $f(x) = x^3 - 9x^2 + 14x + 24$

14. 3, −1, 1, 2

15. −2, 5, −3*i* $f(x) = x^4 - 3x^3 - x^2 - 27x - 90$

16. −4, 4 + *i*

Solve each equation. State the number and type of roots. **17–26. See margin.**

17. $2x^2 + x - 6 = 0$

18. $4x^2 + 1 = 0$

19. $x^3 + 1 = 0$

20. $2x^2 - 5x + 14 = 0$

21. $-3x^2 - 5x + 8 = 0$

22. $8x^3 - 27 = 0$

23. $16x^4 - 625 = 0$

24. $x^3 - 6x^2 + 7x = 0$

25. $x^5 - 8x^3 + 16x = 0$

26. $x^5 + 2x^3 + x = 0$

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function.

27. $f(x) = x^4 - 5x^3 + 2x^2 + 5x + 7$ **0 or 2; 0 or 2; 0, 2, or 4**

28. $f(x) = 2x^3 - 7x^2 - 2x + 12$ **0 or 2; 1; 0 or 2**

29. $f(x) = -3x^5 + 5x^4 + 4x^2 - 8$ **0 or 2; 1; 2 or 4**

30. $f(x) = x^4 - 2x^2 - 5x + 19$ **0 or 2; 0 or 2; 0, 2, or 4**

31. $f(x) = 4x^6 - 5x^4 - x^2 + 24$ **0 or 2; 0 or 2; 2, 4, or 6**

32. $f(x) = -x^5 + 14x^3 + 18x - 36$ **0 or 2; 1; 2 or 4**

Find all of the zeros of each function.

33. $f(x) = x^3 + 7x^2 + 4x - 12$ **−6, −2, 1**

34. $f(x) = x^3 + x^2 - 17x + 15$ **−5, 1, 3**

35. $f(x) = x^4 - 3x^3 - 3x^2 - 75x - 700$ **−4, 7, −5*i*, 5*i***

36. $f(x) = x^4 + 6x^3 + 73x^2 + 384x + 576$ **−3, −3, −8*i*, 8*i***

37. $f(x) = x^4 - 8x^3 + 20x^2 - 32x + 64$

38. $f(x) = x^5 - 8x^3 - 9x$ **−3, 0, 3, −*i*, *i***

Write a polynomial function of least degree with integral coefficients that have the given zeros. **39–44. See margin.**

39. 5, −2, −1

40. −4, −3, 5

41. −1, −1, $2i$

42. −3, 1, $-3i$

43. 0, −5, $3 + i$

44. −2, −3, $4 - 3i$

37. 4, 4, −2*i*, 2*i*

Additional Answers

17. $-2, \frac{3}{2}$; 2 real

18. $-\frac{1}{2}i, \frac{1}{2}i$; 2 imaginary

19. $-1, \frac{1 \pm i\sqrt{3}}{2}$; 1 real, 2 imaginary

20. $\frac{5 \pm i\sqrt{87}}{4}$; 2 imaginary

21. $-\frac{8}{3}, 1$; 2 real

22. $\frac{3}{2}, \frac{-3 \pm 3i\sqrt{3}}{4}$; 1 real, 2 imaginary

23. $-\frac{5}{2}, \frac{5}{2}, -\frac{5}{2}i, \frac{5}{2}i$; 2 real, 2 imaginary

24. $0, 3 + \sqrt{2}, 3 - \sqrt{2}$; 3 real

25. −2, −2, 0, 2, 2; 5 real

26. $0, -i, -i, i, i$; 1 real, 4 imaginary

39. $f(x) = x^3 - 2x^2 - 13x - 10$

40. $f(x) = x^3 + 2x^2 - 23x - 60$

41. $f(x) = x^4 + 2x^3 + 5x^2 + 8x + 4$

42. $f(x) = x^4 + 2x^3 + 6x^2 + 18x - 27$

43. $f(x) = x^4 - x^3 - 20x^2 + 50x$

44. $f(x) = x^4 - 3x^3 - 9x^2 + 77x + 150$