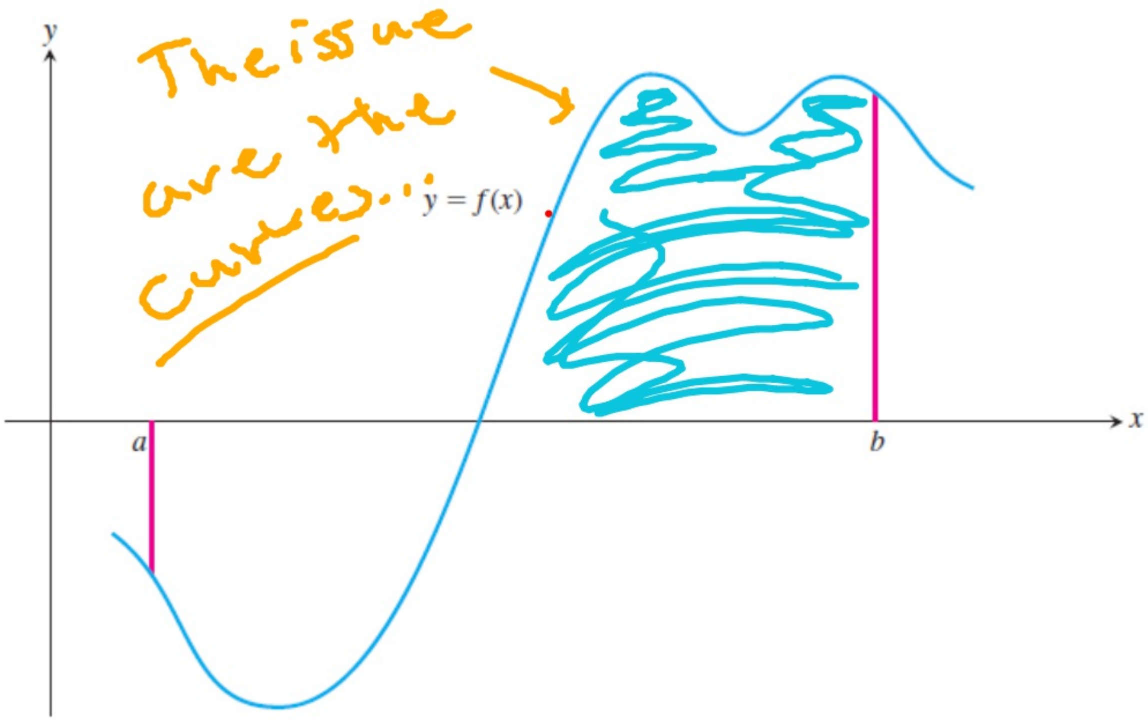
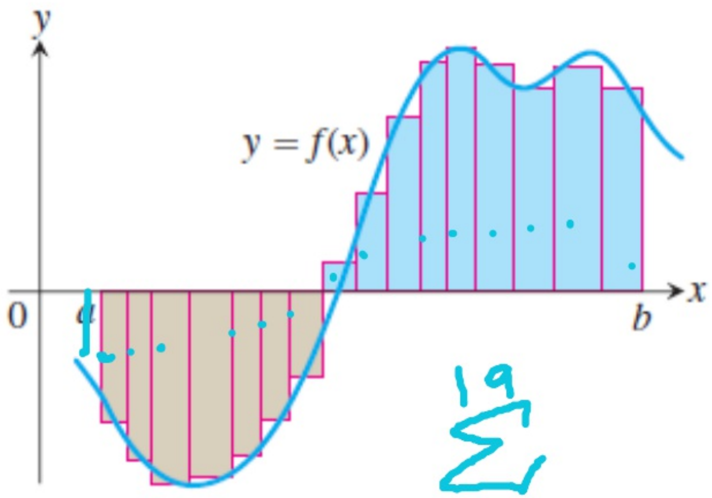
5.2 Definite Integrals

Q: What is the area enclosed between the graph and the x axis in the closed interval [a,b] ?





(a)

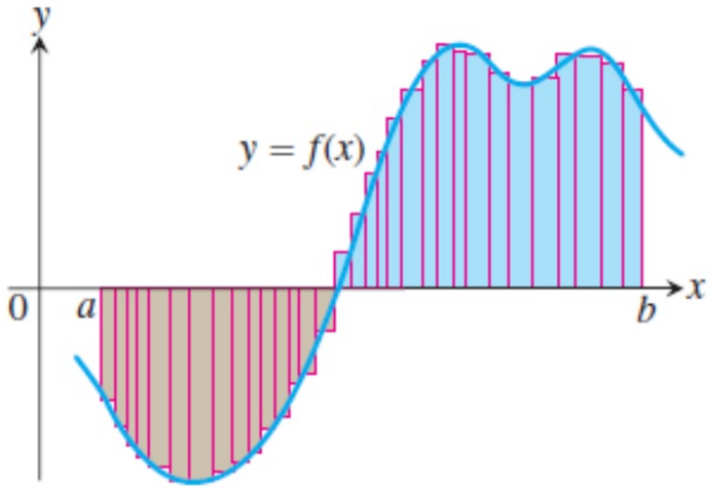
$\sum_{i=1}^{19}$

A: We could break it up into rectangles, then add up all the rectangles...

"height" "base"

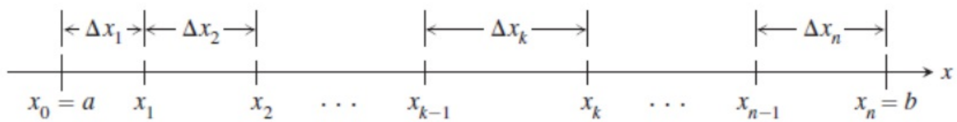
$$S_n = \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k.$$

Riemann sum for f on the interval $[a, b]$.



(b)

The basic idea is, the smaller the partitions, the more accurate the area...

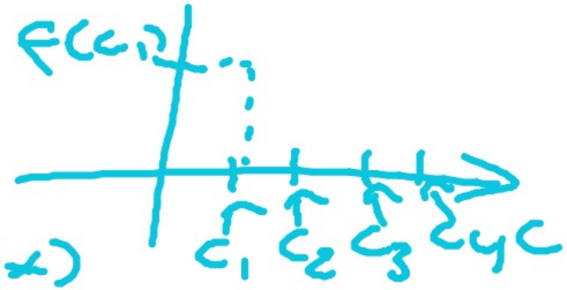


This agruement can be molded into a limit, and is given a new notation altogether;

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x = \int_a^b f(x)\,dx.$$

"height" "base"

$$\sum_{i=1}^{4} f(x)$$



Upper limit of integration

The function is the **integrand.**

Integral sign

$$\int_a^b f(x)\,dx$$

x is the **variable of integration.**

Lower limit of integration

Integral of f from a to b

When you find the value of the integral, you have **evaluated the integral.**

In Exercises 1–6, each c_k is chosen from the kth subinterval of a regular partition of the indicated interval into n subintervals of length Δx. Express the limit as a definite integral.

1. $\lim_{n\to\infty} \sum_{k=1}^{n} c_k^2 \Delta x$, $[0, 2]$ $\int_0^2 x^2\, dx$

2. $\lim_{n\to\infty} \sum_{k=1}^{n} (c_k^2 - 3c_k)\,\Delta x$, $[-7, 5]$ $\int_{-7}^{5} (x^2 - 3x)\, dx$

3. $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{c_k} \Delta x$, $[1, 4]$ $\int_1^4 \frac{1}{x}\, dx$

4. $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{1 - c_k} \Delta x$, $[2, 3]$ $\int_2^3 \frac{1}{1 - x}\, dx$

DEFINITION Area Under a Curve (as a Definite Integral)

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ from a to b** is the integral of f from a to b,

$$A = \int_a^b \underbrace{f(x)\,dx}_{dy \cdot dx}.$$

last semester...

$$\frac{dy}{dx}$$

Note for the eager ones in here;

We will get into the antiderivative, but we will first approach this geometrically.

Also, we will be VERY careful when finding the area

$$\int_a^b f(x)\,dx = (\text{area above the } x\text{-axis}) - (\text{area below the } x\text{-axis}).$$

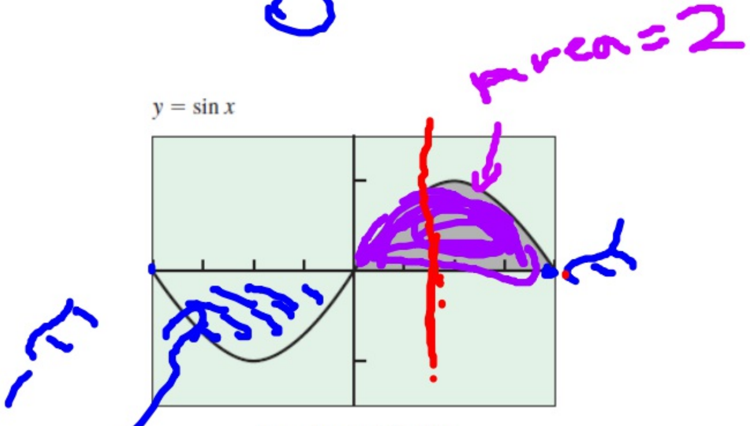
EXPLORATION 1 Finding Integrals by Signed Areas

It is a fact (which we will revisit) that $\int_0^{\pi} \sin x \, dx = 2$ (Figure 5.20). With that information, what you know about integrals and areas, what you know about graphing curves, and sometimes a bit of intuition, determine the values of the following integrals. Give as convincing an argument as you can for each value, based on the graph of the function.

1. $\displaystyle\int_{\pi}^{2\pi} \sin x \, dx = -2$ **2.** $\displaystyle\int_{0}^{2\pi} \sin x \, dx = 0$ **3.** $\displaystyle\int_{0}^{\pi/2} \sin x \, dx = 1$

4. $\displaystyle\int_{0}^{\pi} (2 + \sin x) \, dx$ $2\pi + 2$ **5.** $\displaystyle\int_{0}^{\pi} 2 \sin x \, dx$ 4 **6.** $\displaystyle\int_{2}^{\pi+2} \sin (x - 2) \, dx$

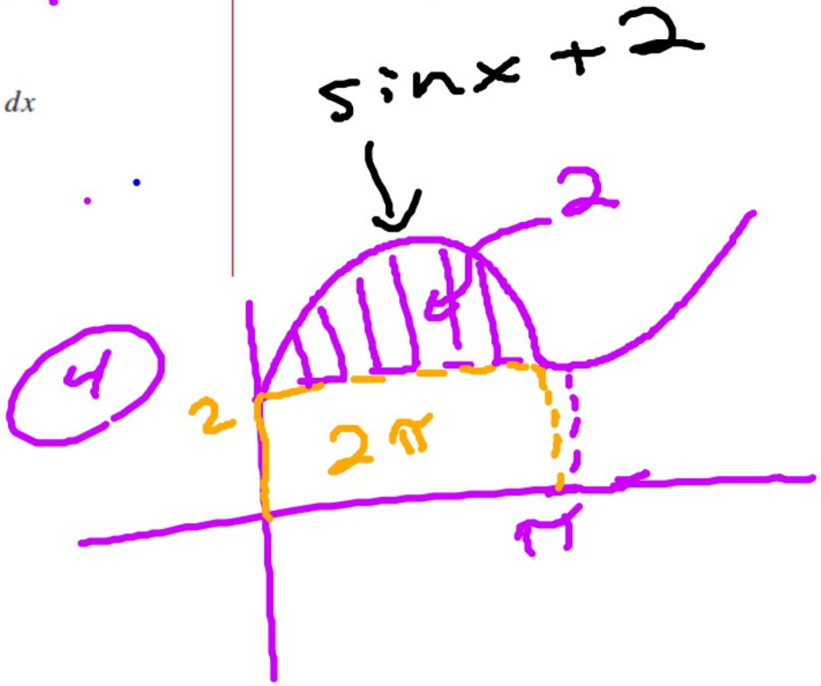
7. $\displaystyle\int_{-\pi}^{\pi} \sin u \, du = 0$ **8.** $\displaystyle\int_{0}^{2\pi} \sin (x/2) \, dx$ **9.** $\displaystyle\int_{0}^{\pi} \cos x \, dx$



$[-\pi, \pi]$ by $[-1.5, 1.5]$

Figure 5.20

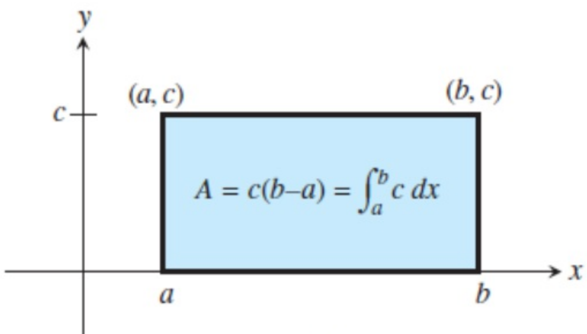
$\int_0^{\pi} \sin x \, dx = 2$. (Exploration 1)



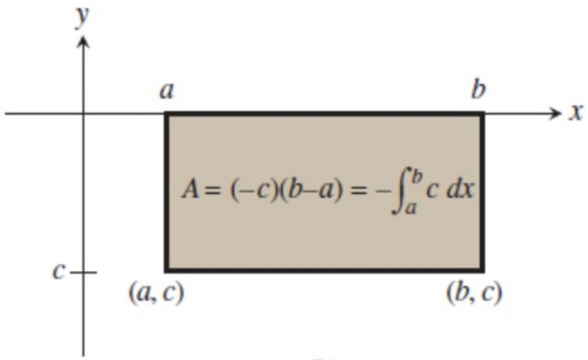
THEOREM 2 The Integral of a Constant

If $f(x) = c$, where c is a constant, on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = \int_a^b c\,dx = c(b - a).$$



(a)



(b)

In Exercises 7–12, evaluate the integral.

7. $\displaystyle\int_{-2}^{1} 5\,dx$ 15

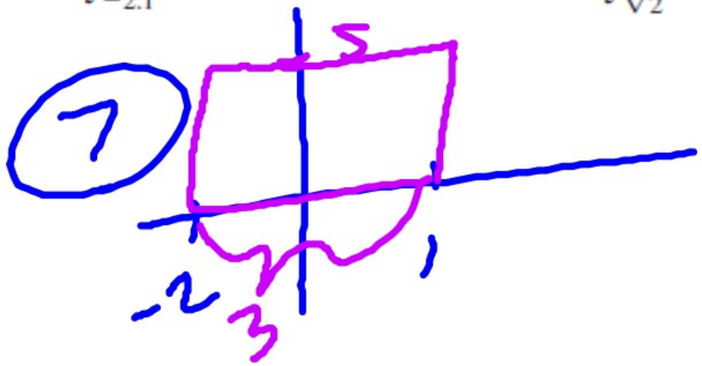
8. $\displaystyle\int_{3}^{7} (-20)\,dx$

9. $\displaystyle\int_{0}^{3} (-160)\,dt$ -480

10. $\displaystyle\int_{-4}^{-1} \frac{\pi}{2}\,d\theta$ $\frac{3\pi}{2}$

11. $\displaystyle\int_{-2.1}^{3.4} 0.5\,ds$

12. $\displaystyle\int_{\sqrt{2}}^{\sqrt{18}} \sqrt{2}\,dr$



In Exercises 13–22, use the graph of the integrand and areas to evaluate the integral.

13. $\int_{-2}^{4} \left(\frac{x}{2} + 3\right) dx$ 21

14. $\int_{1/2}^{3/2} (-2x + 4)\, dx$ 2

15. $\int_{-3}^{3} \sqrt{9 - x^2}\, dx$ $\frac{9\pi}{2}$

16. $\int_{-4}^{0} \sqrt{16 - x^2}\, dx$ 4π

17. $\int_{-2}^{1} |x|\, dx$ $\frac{5}{2}$

18. $\int_{-1}^{1} (1 - |x|)\, dx$ 1

19. $\int_{-1}^{1} (2 - |x|)\, dx$ 3

20. $\int_{-1}^{1} (1 + \sqrt{1 - x^2})\, dx$ $2 + \frac{\pi}{2}$

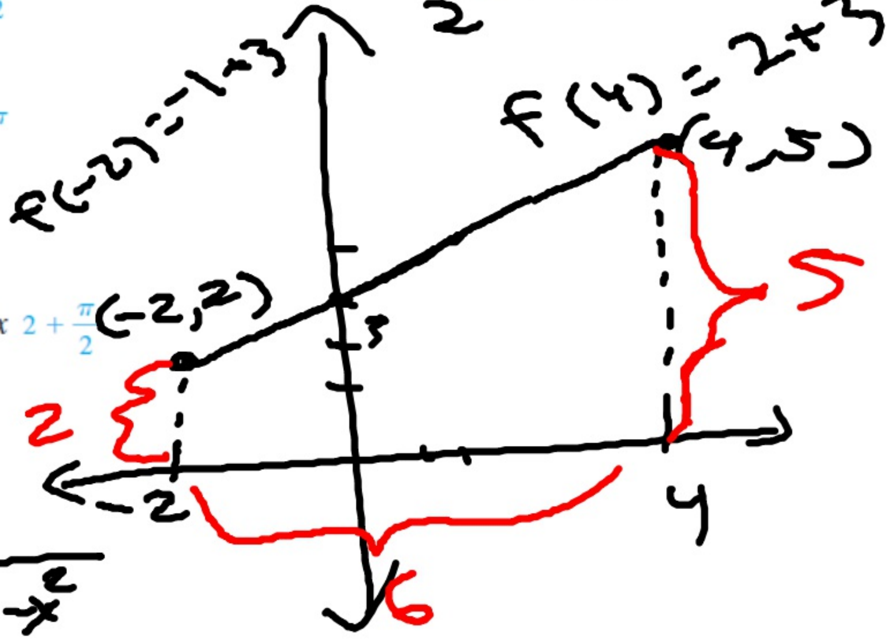
21. $\int_{\pi}^{2\pi} \theta\, d\theta$ $\frac{3\pi^2}{2}$

22. $\int_{\sqrt{2}}^{5\sqrt{2}} r\, dr$ 24

$f(x) = \frac{1}{2}x + 3$

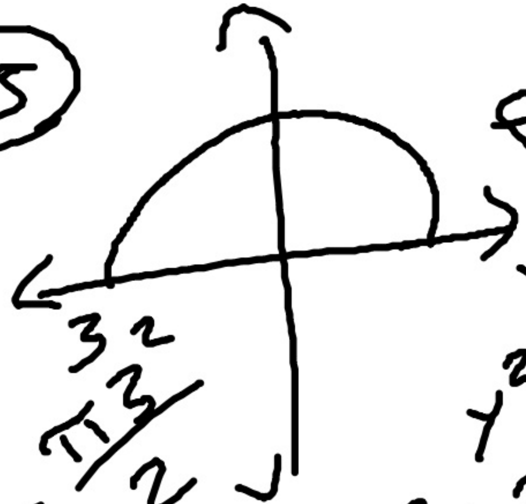
$f(-2) = -1 + 3$

$f(4) = 2 + 3$



$\frac{6(2+5)}{2} = 3(7)$

15



$f(x) = \sqrt{9 - x^2}$

$y = \sqrt{9 - x^2}$

$y^2 = 9 - x^2$

$x^2 + y^2 = 9$

$A = \frac{\pi 3^2}{2}$

In Exercises 13–22, use the graph of the integrand and areas to evaluate the integral.

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18. $\int_{-1}^{1}(1-|x|)\,dx$ 1

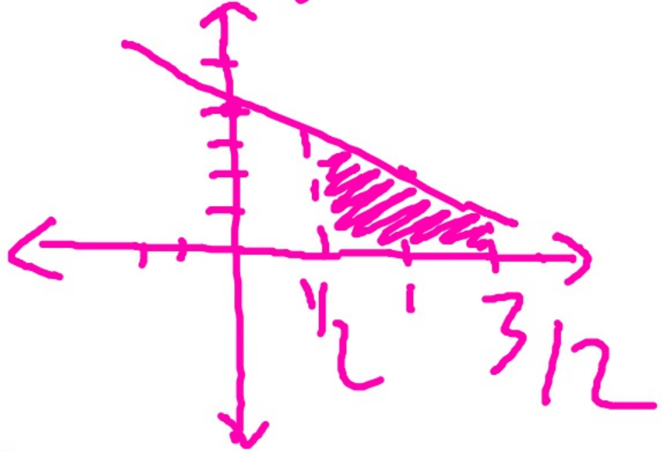
19. $\int_{-1}^{1}(2-|x|)\,dx$ 3

20. $\int_{-1}^{1}\left(1+\sqrt{1-x^2}\right)dx$ $2+\frac{\pi}{2}$

21. $\int_{\pi}^{2\pi}\theta\,d\theta$ $\frac{3\pi^2}{2}$

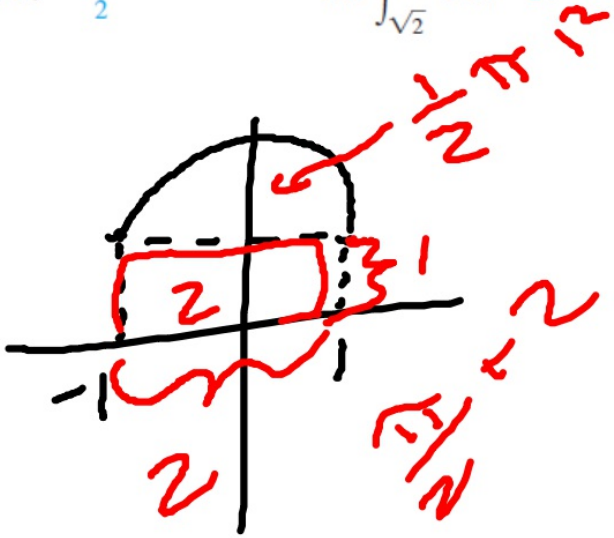
22. $\int_{\sqrt{2}}^{5\sqrt{2}}r\,dr$ 24

14- $f(x)=-2x+4$



$$A=\frac{(3+1)\,1}{2}$$

Area $=2$



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17

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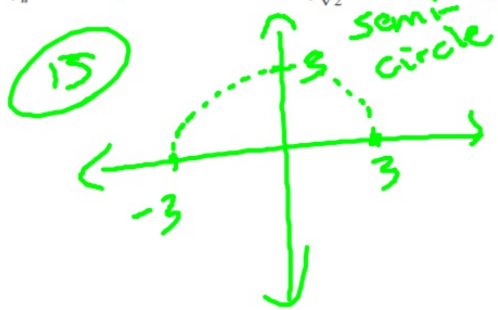
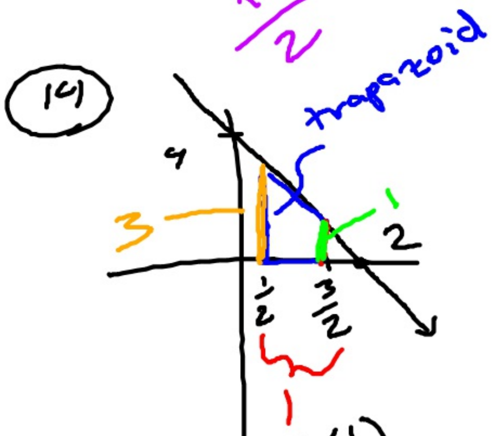


$$\int_{-3}^{3}\sqrt{9-x^2} = \frac{\pi r^2}{2}$$

$r=3$

$$\frac{\pi 3^2}{2} = \frac{9\pi}{2}$$

14



$$A = \frac{1}{2}(1+3)(1) = \frac{4}{2}(1) = 2(1) = 2$$

In Exercises 13–22, use the graph of the integrand and areas to evaluate the integral.

13. $\int_{-2}^{4} \left(\frac{x}{2} + 3\right) dx$ 21

14. $\int_{1/2}^{3/2} (-2x + 4)\, dx$ 2

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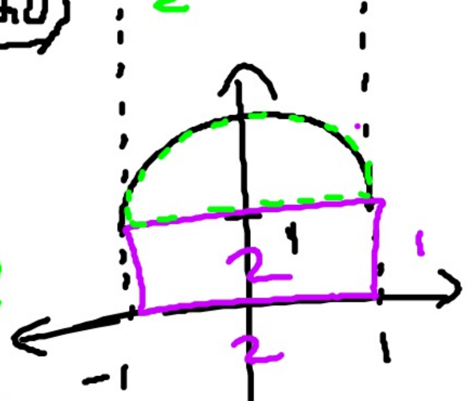
21. $\int_{\pi}^{2\pi} \theta\, d\theta$ $\frac{3\pi^2}{2}$

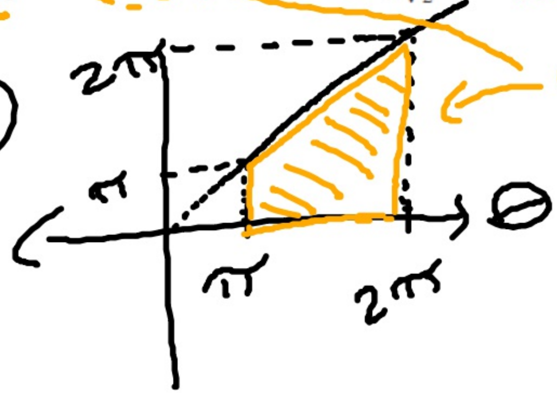
22. $\int_{\sqrt{2}}^{5\sqrt{2}} r\, dr$ 24

20

$\frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$

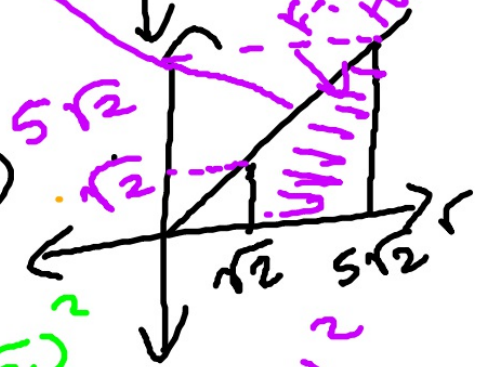


21



Find the area...

22



find this!

$\frac{1}{2}(\sqrt{2})^2$

$\frac{1}{2} \cdot 2$

$= 1$

$\frac{1}{2}(5\sqrt{2})^2$

$\frac{1}{2}(25 \cdot 2) = 25$

In Exercises 23–28, use areas to evaluate the integral.

23. $\int_0^b x\,dx, \quad b > 0$ $\quad \frac{1}{2}b^2$

24. $\int_0^b 4x\,dx, \quad b > 0$ $\quad 2b^2$

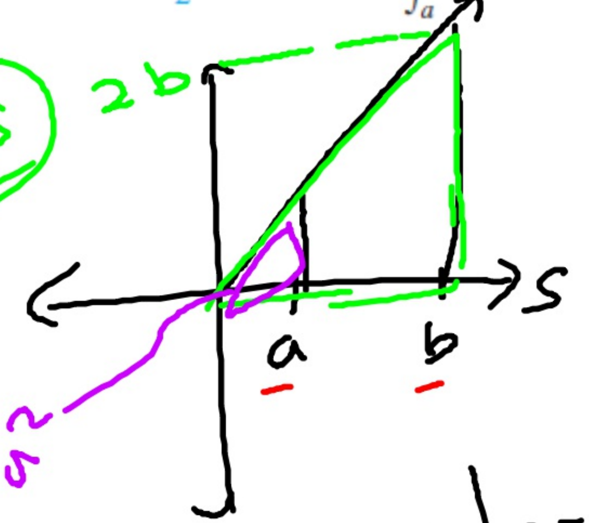
25. $\int_a^b 2s\,ds, \quad 0 < a < b$ $\quad b^2 - a^2$

26. $\int_a^b 3t\,dt, \quad 0 < a < b$ $\quad \frac{3}{2}(b^2 - a^2)$

27. $\int_a^{2a} x\,dx, \quad a > 0$ $\quad \frac{3}{2}a^2$

28. $\int_a^{\sqrt{3}a} x\,dx, \quad a > 0$ $\quad a^2$

25

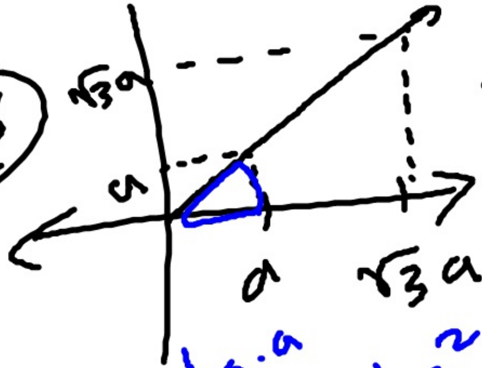


$\frac{1}{2}b \cdot h$

$\frac{1}{2}b \cdot 2b$

$2\,\frac{1}{2}b^2 = b^2$

28



$\frac{1}{2}a \cdot a = \frac{1}{2}a^2$

$\frac{1}{2}\sqrt{3}a \cdot \sqrt{3}a$

$= \frac{3a^2}{2}$

$\frac{3a^2}{2} - \frac{a^2}{2} = a^2$

In Exercises 23–28, use areas to evaluate the integral.

23. $\int_0^b x\,dx, \quad b > 0$ $\quad \frac{1}{2}b^2$

24. $\int_0^b 4x\,dx, \quad b > 0$ $\quad 2b^2$

25. $\int_a^b 2s\,ds, \quad 0 < a < b$ $\quad b^2 - a^2$

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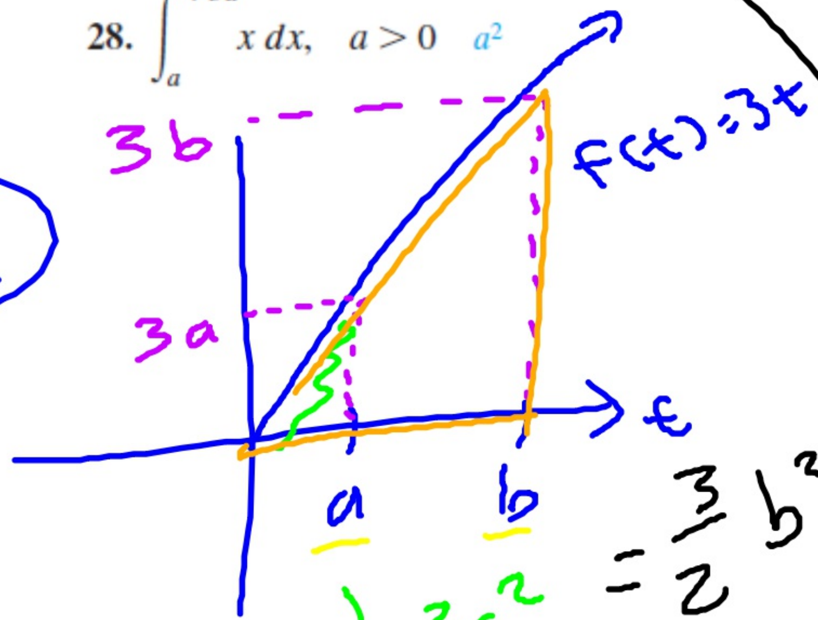
27. $\int_a^{2a} x\,dx, \quad a > 0$ $\quad \frac{3}{2}a^2$

28. $\int_a^{\sqrt{3}a} x\,dx, \quad a > 0$ $\quad a^2$

26



$$\frac{1}{2}3b^2 - \frac{1}{2}3a^2 = \frac{3}{2}b^2 - \frac{3}{2}a^2$$

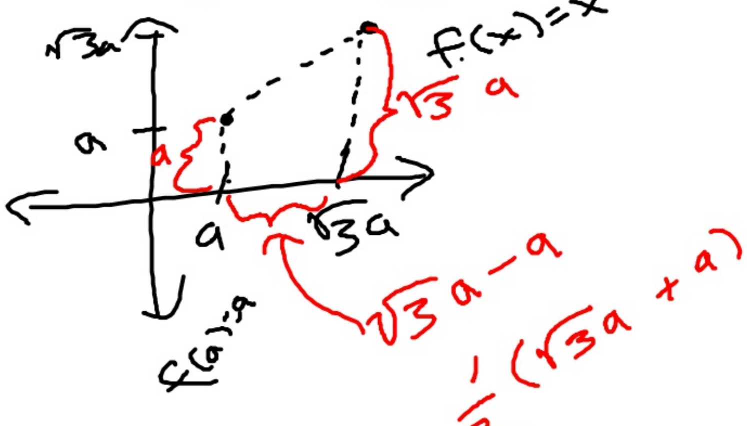
In Exercises 23–28, use areas to evaluate the integral.

23. $\int_0^b x\,dx$, $\quad b > 0 \quad \frac{1}{2}b^2$ **24.** $\int_0^b 4x\,dx$, $\quad b > 0 \quad 2b^2$

25. $\int_a^b 2s\,ds$, $\quad 0 < a < b \quad b^2 - a^2$ **26.** $\int_a^b 3t\,dt$, $\quad 0 < a < b \quad \frac{3}{2}(b^2 - a^2)$

$f(t) = 3t$

27. $\int_a^{2a} x\,dx$, $\quad a > 0 \quad \frac{3}{2}a^2$ **28.** $\int_a^{\sqrt{3}a} x\,dx$, $\quad a > 0 \quad a^2$



$$A = \frac{1}{2}(\sqrt{3}a + a)(\sqrt{3}a - a)$$

$$\frac{1}{2}(3a^2 - a^2) = \frac{1}{2}(2a^2)$$

$$= a^2$$

In Exercises 29–32, express the desired quantity as a definite integral and evaluate the integral using Theorem 2.

29. Find the distance traveled by a train moving at 87 mph from 8:00 A.M. to 11:00 A.M.

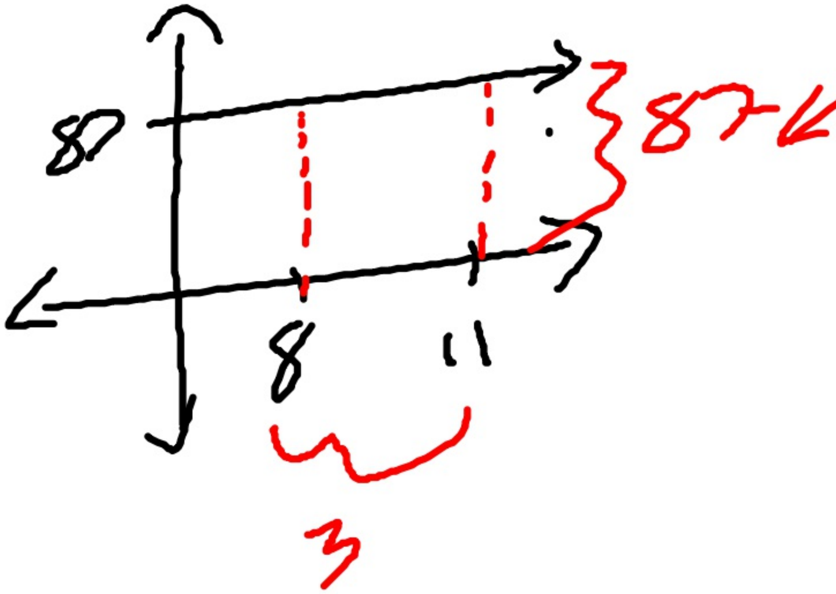
$y=87$

87×3

30. Find the output from a pump producing 25 gallons per minute during the first hour of its operation.

31. Find the calories burned by a walker burning 300 calories per hour between 6:00 P.M. and 7:30 P.M.

32. Find the amount of water lost from a bucket leaking 0.4 liters per hour between 8:30 A.M. and 11:00 A.M.



Answers:

29. $\int_8^{11} 87\,dt = 261$ miles

30. $\int_0^{60} 25\,dt = 1500$ gallons

31. $\int_6^{7.5} 300\,dt = 450$ calories

32. $\int_{8.5}^{11} 0.4\,dt = 1$ liter