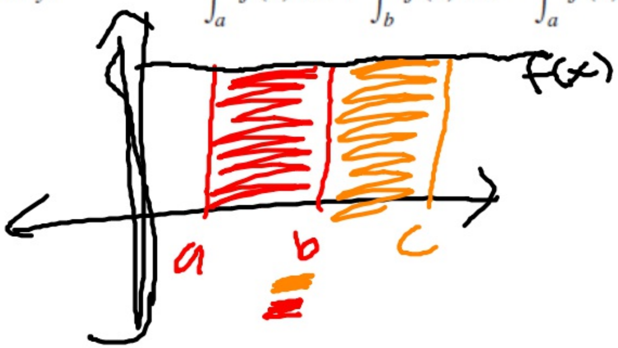
5.3 Definite Integrals and Antiderivatives

Table 5.3 Rules for Definite Integrals

1. *Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A definition
2. *Zero:*	$\int_a^a f(x)\,dx = 0$	Also a definition
3. *Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any number k
	$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$	$k = -1$
4. *Sum and Difference:*	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
5. *Additivity:*	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	

$+ \longrightarrow$

$\longleftarrow -$



EXAMPLE 1 Using the Rules for Definite Integrals

Suppose

+

$$\int_{-1}^{1} f(x)\,dx = 5, \quad \int_{1}^{4} f(x)\,dx = -2, \quad \text{and} \quad \int_{-1}^{1} h(x)\,dx = 7.$$

Find each of the following integrals, if possible.

(a) $\displaystyle\int_{4}^{1} f(x)\,dx$ **(b)** $\displaystyle\int_{-1}^{4} f(x)\,dx$ **(c)** $\displaystyle\int_{-1}^{1} [2f(x) + 3h(x)]\,dx$

(d) $\displaystyle\int_{0}^{1} f(x)\,dx$ **(e)** $\displaystyle\int_{-2}^{2} h(x)\,dx$ **(f)** $\displaystyle\int_{-1}^{4} [f(x) + h(x)]\,dx$

SOLUTION

(a) $\displaystyle\int_{4}^{1} f(x)\,dx = -\int_{1}^{4} f(x)\,dx = -(-2) = 2$

(b) $\displaystyle\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx = 5 + (-2) = 3$

(c) $\displaystyle\int_{-1}^{1} [2f(x) + 3h(x)]\,dx = 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx = 2(5) + 3(7) = 31$

(d) Not enough information given. (We cannot assume, for example, that integrating over half the interval would give half the integral!)

(e) Not enough information given. (We have no information about the function h outside the interval $[-1, 1]$.)

(f) Not enough information given (same reason as in part (e)). ***Now try Exercise 1.***

The exercises in this section are designed to reinforce your understanding of the definite integral from the algebraic and geometric points of view. For this reason, you should not use the numerical integration capability of your calculator (NINT) except perhaps to support an answer.

1. Suppose that f and g are continuous functions and that

$$\int_1^2 f(x)\,dx = -4, \quad \int_1^5 f(x)\,dx = 6, \quad \int_1^5 g(x)\,dx = 8.$$

Use the rules in Table 5.3 to find each integral.

(a) $\int_2^2 g(x)\,dx$ 0 **(b)** $\int_5^1 g(x)\,dx$ -8

(c) $\int_1^2 3f(x)\,dx$ -12 **(d)** $\int_2^5 f(x)\,dx$ 10

(e) $\int_1^5 [f(x) - g(x)]\,dx$ -2 **(f)** $\int_1^5 [4f(x) - g(x)]\,dx$ 16

2. Suppose that f and h are continuous functions and that

$$\int_1^9 f(x)\,dx = -1, \quad \int_7^9 f(x)\,dx = 5, \quad \int_7^9 h(x)\,dx = 4.$$

Use the rules in Table 5.3 to find each integral.

(a) $\int_1^9 -2f(x)\,dx$ 2 **(b)** $\int_7^9 [f(x) + h(x)]\,dx$ 9

(c) $\int_7^9 [2f(x) - 3h(x)]\,dx$ -2 **(d)** $\int_9^1 f(x)\,dx$ 1

(e) $\int_1^7 f(x)\,dx$ -6 **(f)** $\int_9^7 [h(x) - f(x)]\,dx$ 1

3. Suppose that $\int_1^2 f(x)\,dx = 5$. Find each integral.

(a) $\int_1^2 f(u)\,du$ 5 **(b)** $\int_1^2 \sqrt{3}\,f(z)\,dz$ $5\sqrt{3}$

(c) $\int_2^1 f(t)\,dt$ -5 **(d)** $\int_1^2 [-f(x)]\,dx$ -5

4. Suppose that $\int_{-3}^0 g(t)\,dt = \sqrt{2}$. Find each integral.

(a) $\int_0^{-3} g(t)\,dt$ $-\sqrt{2}$ **(b)** $\int_{-3}^0 g(u)\,du$ $\sqrt{2}$

(c) $\int_{-3}^0 [-g(x)]\,dx$ $-\sqrt{2}$ **(d)** $\int_{-3}^0 \frac{g(r)}{\sqrt{2}}\,dr$ 1

5. Suppose that f is continuous and that

$$\int_0^3 f(z)\,dz = 3 \quad \text{and} \quad \int_0^4 f(z)\,dz = 7.$$

Find each integral.

(a) $\int_3^4 f(z)\,dz$ 4 **(b)** $\int_4^3 f(t)\,dt$ -4

6. Suppose that h is continuous and that

$$\int_{-1}^1 h(r)\,dr = 0 \quad \text{and} \quad \int_{-1}^3 h(r)\,dr = 6.$$

Find each integral.

(a) $\int_1^3 h(r)\,dr$ 6 **(b)** $-\int_3^1 h(u)\,du$ 6

skip #7-14!

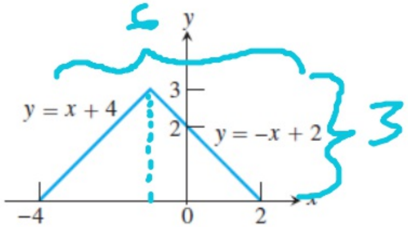
DEFINITION **Average (Mean) Value**

If f is integrable on $[a, b]$, its **average (mean) value** on $[a, b]$ is

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

In Exercises 15–18, find the average value of the function on the interval without integrating, by appealing to the geometry of the region between the graph and the x-axis.

15. $f(x) = \begin{cases} x+4, & -4 \le x \le -1, \\ -x+2, & -1 < x \le 2, \end{cases}$ on $[-4, 2]$ $\frac{3}{2}$



$A = \frac{1}{2}(6)(3)$

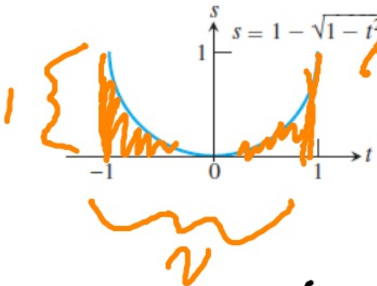
$A = 9 = \int_a^b f(x)dx$

$A = 9$

$\frac{1}{6} = \frac{1}{b-a}$

$av(f) = 9/6$

16. $f(t) = 1 - \sqrt{1-t^2}$, $[-1, 1]$ $\frac{4-\pi}{4}$



rectangle – semi circle

$2(1) - \frac{\pi r^2}{2} = 2 - \frac{\pi}{2} = \int_a^b f(x)dx$

$\frac{1}{2} = \frac{1}{b-a}$

$av(f) = \frac{1}{2}\left(2 - \frac{\pi}{2}\right)$

$= 1 - \frac{\pi}{4}$

17. $f(t) = \sin t$, $[0, 2\pi]$ 0

18. $f(\theta) = \tan\theta$, $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ 0

If all went well in Exploration 2, you concluded that the derivative with respect to x of the integral of f from a to x is simply f. Specifically,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

This means that the integral is an *antiderivative* of f, a fact we can exploit in the following way.

If F is any antiderivative of f, then

$$\int_a^x f(t)\,dt = F(x) + C$$

for some constant C. Setting x in this equation equal to a gives

$$\int_a^a f(t)\,dt = F(a) + C$$

$$0 = F(a) + C$$

$$C = -F(a).$$

Putting it all together,

$$\int_a^x f(t)\,dt = F(x) - F(a).$$

This is kinda special, and I'll explain why later....

differentiation | integration

$\frac{dy}{dx}$ | $dy \cdot dx$

$$\int_a^x f(t)\,dt = F(x) - F(a).$$

In Exercises 19–30, evaluate the integral using antiderivatives, as in Example 4.

19. $\int_{\pi}^{2\pi} \sin x\,dx$

$-\cos(2\pi) + \cos\pi = -2$

20. $\int_0^{\pi/2} \cos x\,dx = \sin\left(\frac{\pi}{2}\right) - \sin(0)$

$1 - 0 = 1$

21. $\int_0^1 e^x\,dx$

22. $\int_0^{\pi/4} \sec^2 x\,dx$

23. $\int_1^4 2x\,dx$

24. $\int_{-1}^2 3x^2\,dx$

25. $\int_{-2}^6 5\,dx$

26. $\int_3^7 8\,dx$

$8x$

$= 8(7) - 8(3)$

$56 - 24 = 32$

27. $\int_{-1}^1 \frac{1}{1+x^2}\,dx$

$\tan^{-1}(1) - \tan^{-1}(-1) = \pi/2$

28. $\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx$

29. $\int_1^e \frac{1}{x}\,dx$

30. $\int_1^4 -x^{-2}\,dx$

In Exercises 19–30, evaluate the integral using antiderivatives, as in Example 4.

19. $\int_{\pi}^{2\pi} \sin x \, dx$ $-\cos(2\pi) + \cos \pi = -2$

20. $\int_{0}^{\pi/2} \cos x \, dx$ $\sin(\pi/2) - \sin 0 = 1$

21. $\int_{0}^{1} e^x \, dx$ $e^1 - e^0 = e - 1$

22. $\int_{0}^{\pi/4} \sec^2 x \, dx$ $\tan(\pi/4) - \tan 0 = 1$

23. $\int_{1}^{4} 2x \, dx$ $4^2 - 1^2 = 15$

24. $\int_{-1}^{2} 3x^2 \, dx$ $2^3 - (-1)^3 = 9$

25. $\int_{-2}^{6} 5 \, dx$ $5(6) - 5(-2) = 40$

26. $\int_{3}^{7} 8 \, dx$ $8(7) - 8(3) = 32$

27. $\int_{-1}^{1} \frac{1}{1+x^2} \, dx$ $\tan^{-1}(1) - \tan^{-1}(-1) = \pi/2$

28. $\int_{0}^{1/2} \frac{1}{\sqrt{1-x^2}} \, dx$ $\sin^{-1}(1/2) - \sin^{-1}(0) = \pi/6$

29. $\int_{1}^{e} \frac{1}{x} dx$ $\ln e - \ln 1 = 1$

30. $\int_{1}^{4} -x^{-2} \, dx$ $4^{-1} - 1^{-1} = -3/4$

$$\int_a^x f(t)\,dt = F(x) - F(a).$$

DEFINITION Average (Mean) Value

If f is integrable on $[a, b]$, its **average (mean) value** on $[a, b]$ is

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

NOW, use both!

In Exercises 31–36, find the average value of the function on the interval, using antiderivatives to compute the integral.

31. $y = \sin x, \quad [0, \pi]$ $\quad 2/\pi$

32. $y = \dfrac{1}{x}, \quad [e, 2e]$ $\quad \dfrac{\ln 2}{e}$

33. $y = \sec^2 x, \quad \left[0, \dfrac{\pi}{4}\right]$ $\quad 4/\pi$

34. $y = \dfrac{1}{1+x^2}, \quad [0\,,1]$ $\quad \pi/4$

35. $y = 3x^2 + 2x, \quad [-1, 2]$ $\quad 4$

36. $y = \sec x \tan x, \quad \left[0, \dfrac{\pi}{3}\right]$ $\quad 3/\pi$