5.4 The Fundamental Theorem of Calculus

THEOREM 4 The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a, b]$, then the function

$$F(x) = \int_a^x f(t)\, dt$$

has a derivative at every point x in $[a, b]$, and

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\, dt = f(x).$$

This says two things;

1) ALL functions have an antiderivative.

2) derivatives and integration are inverses to each other.

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x). \qquad (1)$$

EXAMPLE 1 Applying the Fundamental Theorem

Find

$$\frac{d}{dx}\int_{-\pi}^{x} \cos t\,dt \quad \text{and} \quad \frac{d}{dx}\int_0^x \frac{1}{1+t^2}\,dt$$

by using the Fundamental Theorem.

SOLUTION

$$\frac{d}{dx}\int_{-\pi}^{x} \cos t\,dt = \cos x \qquad \text{Eq. 1 with } f(t) = \cos t$$

$$\frac{d}{dx}\int_0^x \frac{1}{1+t^2}\,dt = \frac{1}{1+x^2}. \qquad \text{Eq. 1 with } f(t) = \frac{1}{1+t^2}$$

Now try Exercise 3.

WAIT...doesn't it matter where the integration starts for these...?

Apply the definition of the derivative directly to the function F. That is,

$$\frac{dF}{dx} = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

$$= \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$

$$= \lim_{h\to 0} \left[\frac{1}{h}\int_x^{x+h} f(t)\,dt\right].$$

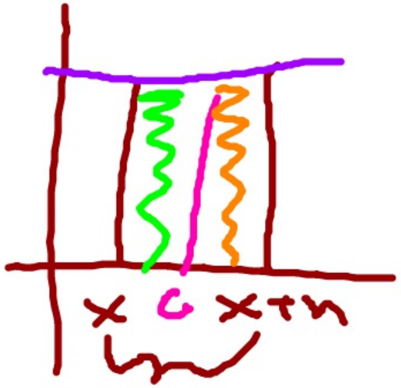
$F(x) = \int_a^x f(t)\,dt$

This must be constant;

$\int_a^{x+h} f(t)dt = \int_a^{x} f(x)dt + \int_x^{x+h} f(t)dt$

A: nope.

Rules for integrals, Section 5.3



h approaches zero!!

$$\frac{dF}{dx} = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

$$= \lim_{h\to 0} f(c), \quad \text{where } c \text{ lies between } x \text{ and } x+h.$$

What happens to c as h goes to zero? As $x + h$ gets closer to x, it carries c along with it like a bead on a wire, forcing c to approach x. Since f is continuous, this means that $f(c)$ approaches $f(x)$:

$$\lim_{h\to 0} f(c) = f(x).$$

BTW, "a" is a constant. BUT what if it's not...?

Also, what if there's some weird stuff going on in the limits of intergration...?

In Exercises 1–20, find dy/dx.

1. $y = \int_0^x (\sin^2 t)\, dt$ $\quad \sin^2 x$

constant!!

2. $y = \int_2^x (3t + \cos t^2)\, dt$ $\quad 3x + \cos x^2$

3. $y = \int_0^x (t^3 - t)^5\, dt$ $\quad (x^3 - x)^5$

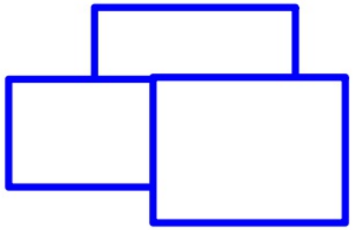
4. $y = \int_{-2}^x \sqrt{1 + e^{5t}}\, dt$ $\quad \sqrt{1 + e^{5x}}$

5. $y = \int_2^x (\tan^3 u) du$ $\quad \tan^3 x$

6. $y = \int_4^x e^u \sec u\, du$ $\quad e^x \sec x$

7. $y = \int_7^x \frac{1+t}{1+t^2}\, dt$ $\quad \frac{1+x}{1+x^2}$

8. $y = \int_{-\pi}^x \frac{2 - \sin t}{3 + \cos t}\, dt$ $\quad \frac{2 - \sin x}{3 + \cos x}$



EXAMPLE 2 The Fundamental Theorem with the Chain Rule

Find dy/dx if $y = \int_1^{x^2} \cos t\, dt$.

$\frac{d}{dx} f(g(x))$

SOLUTION

The upper limit of integration is not x but x^2. This makes y a composite of

$$y = \int_1^u \cos t\, dt \quad \text{and} \quad u = x^2.$$

$= f'(g(x)) \cdot g'(x)$

We must therefore apply the Chain Rule when finding dy/dx.

$g' = 2x$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= \left(\frac{d}{du} \int_1^u \cos t\, dt \right) \cdot \frac{du}{dx} \\
&= \cos u \cdot \frac{du}{dx} \\
&= \cos(x^2) \cdot 2x \\
&= 2x \cos x^2
\end{aligned}$$

$y' = \cos u$

Now try Exercise 9.

9. $y = \int_0^{x^2} e^{t^2}\,dt$ $\quad 2xe^{x^4}$

10. $y = \int_6^{x^2} \cot 3t\,dt$ [redacted]

11. $y = \int_2^{5x} \frac{\sqrt{1+u^2}}{u}\,du$ $\quad \frac{\sqrt{1+25x^2}}{x}$

12. $y = \int_\pi^{\pi - x} \frac{1+\sin^2 u}{1+\cos^2 u}\,du$

13. $y = \int_x^6 \ln(1+t^2)\,dt$ $\quad -\ln(1+x^2)$

14. $y = \int_x^7 \sqrt{2t^4+t+1}\,dt$ $\quad -\sqrt{2x^4+x+1}$

[redacted]

$u = x^2$

$\frac{du}{dx} = 2x$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

⑨ $y = \int_0^u e^{t^2}$

$\frac{dy}{du} = e^{u^2}$

$\frac{dy}{dx} = e^{x^4} \cdot 2x$

11. $y = \int_2^{5x} \frac{\sqrt{1+u^2}}{u}\,du$

$\frac{\sqrt{1+25x^2}}{x}$

$u = 5x$

$\frac{du}{dx} = 5$

$\frac{dy}{du} = \frac{\sqrt{1+u^2}}{u}$

$\frac{dy}{dx} = \frac{\sqrt{1+(5x)^2}}{\cancel{5}x} \cdot \cancel{5}$

13. $y = \int_x^6 \ln(1 + t^2)\, dt$

$-\ln(1 + x^2)$

$y = -\int_6^x \ln(1+t^2)$

$= -\ln(1+x^2)$

$$= -3x \sin x$$

(b) $$\frac{d}{dx}\int_{2x}^{x^2} \frac{1}{2+e^t}\,dt = \frac{d}{dx}\left(\int_0^{x^2} \frac{1}{2+e^t}\,dt - \int_0^{2x} \frac{1}{2+e^t}\,dt\right)$$

$$= \frac{1}{2+e^{x^2}}\frac{d}{dx}(x^2) - \frac{1}{2+e^{2x}}\frac{d}{dx}(2x) \qquad \text{Chain Rule}$$

$$= \frac{1}{2+e^{x^2}} \cdot 2x - \frac{1}{2+e^{2x}} \cdot 2$$

$$= \frac{2x}{2+e^{x^2}} - \frac{2}{2+e^{2x}}$$

Now try Exercise 19.

15. $y = \int_{x^3}^{5} \frac{\cos t}{t^2+2}\,dt$ $\quad -\frac{3x^2\cos x^3}{x^6+2}$

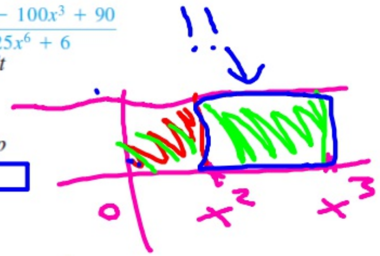
16. $y = \int_{5x^2}^{25} \frac{t^2-2t+9}{t^3+6}\,dt$ $\quad \frac{250x^5-100x^3+90}{125x^6+6}$

17. $y = \int_{\sqrt{x}}^{0} \sin(r^2)\,dr$ $\quad -\frac{\sin x}{2\sqrt{x}}$

18. $y = \int_{3x^2}^{10} \ln(2+p^2)\,dp$

19. $y = \int_{x^2}^{x^3} \cos(2t)\,dt$ $\quad 3x^2\cos(2x^3) - 2x\cos(2x^2)$

20. $y = \int_{\sin x}^{\cos x} t^2\,dt$



15 $\quad \int_{x^3}^{5} \frac{\cos t}{t^2+2} = -\int_{5}^{x^3} \frac{\cos t}{t^2+2}$

19 $\quad \int_{x^2}^{x^3} \cos(2t)\,dt = \int_{0}^{x^3} \cos(2t)\,dt - \int_{0}^{x^2} \cos 2t\,dt$

$u = x^3$

$\frac{du}{dx} = 3x^2$

$y = \int^{u} \cos(2t)$

$= \cos(2u)$

$\frac{dy}{dx} = 3x^2\cos(2x^3)$

$u = x^2$

$\frac{du}{dx} = 2x$

$y = \int_{0}^{u} \cos 2t\,dt$

$\frac{dy}{du} = \cos 2u$

$2x\cos 2x^2$

$3x^2\cos 2x^3 - 2x\cos 2x^2$

EXAMPLE 4 Constructing a Function with a Given Derivative and Value

Find a function $y = f(x)$ with derivative

$$\frac{dy}{dx} = \tan x$$

that satisfies the condition $f(3) = 5$.

SOLUTION

The Fundamental Theorem makes it easy to construct a function with derivative $\tan x$:

$$y = \int_3^x \tan t \, dt.$$

Since $y(3) = 0$, we have only to add 5 to this function to construct one with derivative $\tan x$ whose value at $x = 3$ is 5:

$$f(x) = \int_3^x \tan t \, dt + 5.$$

Now try Exercise 25.

In Exercises 21–26, construct a function of the form $y = \int_a^x f(t)\,dt + C$ that satisfies the given conditions.

relations	functions
$\int_a^x f(x)$	$F(x)$
y	$f(x)$
$\frac{dy}{dx}$	$f'(x)$

21. $\dfrac{dy}{dx} = \sin^3 x$, and $y = 0$ when $x = 5$. $\quad y = \int_5^x \sin^3 t\,dt$

22. $\dfrac{dy}{dx} = e^x \tan x$, and $y = 0$ when $x = 8$. $\quad y = \int_8^x e^t \tan t\,dt$

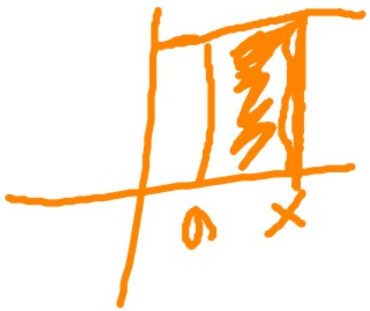
23. $\dfrac{dy}{dx} = \ln(\sin x + 5)$, and $y = 3$ when $x = 2$.

$y = \int_2^x \ln(\sin t + 5)\,dt + 3$

$$F(x) = \int_2^x \ln(\sin t + 5)\,dt + C$$

$$F(2) = \int_2^2 \ln(\sin t + 5)\,dt + C$$

$$3 = 0 + C$$

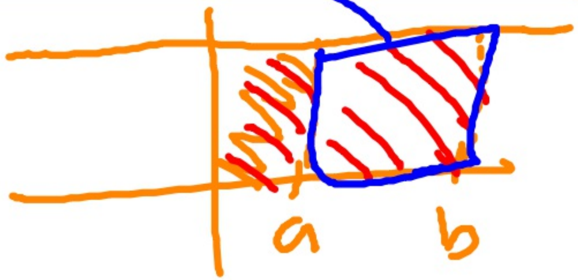


THEOREM 4 (continued) **The Fundamental Theorem of Calculus, Part 2**

If f is continuous at every point of $[a, b]$, and if F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This part of the Fundamental Theorem is also called the **Integral Evaluation Theorem.**



EXAMPLE 5 Evaluating an Integral

Evaluate $\int_{-1}^{3} (x^3 + 1)\,dx$ using an antiderivative.

SOLUTION

Solve Analytically A simple antiderivative of $x^3 + 1$ is $(x^4/4) + x$. Therefore,

$$\int_{-1}^{3}\left(x^3 + 1\right)dx = \left[\frac{x^4}{4} + x\right]_{-1}^{3}$$

$$= \left(\frac{81}{4} + 3\right) - \left(\frac{1}{4} - 1\right)$$

$$= 24.$$

Support Numerically NINT $(x^3 + 1, x, -1, 3) = 24.$ *Now try Exercise 29.*

In Exercises 27–40, evaluate each integral using Part 2 of the Fundamental Theorem. Support your answer with NINT if you are unsure.

What is this?

27. $\int_{1/2}^{3} \left(2 - \frac{1}{x}\right) dx$ $\quad 5 - \ln 6 \approx 3.208$

28. $\int_{2}^{-1} 3^x \, dx$ $\quad -\frac{26}{3 \ln 3} \approx -7.889$

29. $\int_{0}^{1} (x^2 + \sqrt{x}) \, dx$ $\quad 1$

30. $\int_{0}^{5} x^{3/2} \, dx$ $\quad 10\sqrt{5} \approx 22.361$

31. $\int_{1}^{32} x^{-6/5} \, dx$ $\quad \frac{5}{2}$

32. $\int_{-2}^{-1} \frac{2}{x^2} \, dx$ $\quad 1$

33. $\int_{0}^{\pi} \sin x \, dx$ $\quad 2$

34. $\int_{0}^{\pi} (1 + \cos x) \, dx$ $\quad \pi$

35. $\int_{0}^{\pi/3} 2 \sec^2 \theta \, d\theta$ $\quad 2\sqrt{3}$

36. $\int_{\pi/6}^{5\pi/6} \csc^2 \theta \, d\theta$ $\quad 2\sqrt{3}$

37. $\int_{\pi/4}^{3\pi/4} \csc x \cot x \, dx$ $\quad 0$

38. $\int_{0}^{\pi/3} 4 \sec x \tan x \, dx$ $\quad 4$

39. $\int_{-1}^{1} (r + 1)^2 \, dr$ $\quad \frac{8}{3}$

40. $\int_{0}^{4} \frac{1 - \sqrt{u}}{\sqrt{u}} \, du$ $\quad 0$

$\frac{1}{2}$

$\int_0^4 \frac{1}{\sqrt{u}} - \frac{\sqrt{u}}{\sqrt{u}} \, du$

$u^{1/2}$

$= \int_0^4 u^{-1/2} - 1 \, du$

power rule!

$2u^{1/2} - u \Big|_0^4$

$\sqrt{4}$

$(2(4)^{1/2} - 4) - (0 - 0)$

$(2(2) - 4) = 0$

In Exercises 27–40, evaluate each integral using Part 2 of the Fundamental Theorem. Support your answer with NINT if you are unsure.

27. $\int_{1/2}^{3} \left(2 - \frac{1}{x}\right) dx$ $5 - \ln 6 \approx 3.208$

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31. $\int_{1}^{32} x^{-6/5} \, dx$ $\frac{5}{2}$

32. $\int_{-2}^{-1} \frac{2}{x^2} \, dx$ 1

33. $\int_{0}^{\pi} \sin x \, dx$ 2

34. $\int_{0}^{\pi} (1 + \cos x) \, dx$ π

35. $\int_{0}^{\pi/3} 2 \sec^2\theta \, d\theta$ $2\sqrt{3}$

36. $\int_{\pi/6}^{5\pi/6} \csc^2\theta \, d\theta$ $2\sqrt{3}$

37. $\int_{\pi/4}^{3\pi/4} \csc x \cot x \, dx$ 0

38. $\int_{0}^{\pi/3} 4 \sec x \tan x \, dx$ 4

39. $\int_{-1}^{1} (r + 1)^2 \, dr$ $\frac{8}{3}$

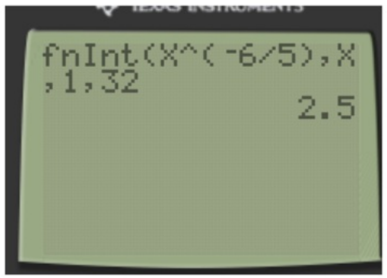
40. $\int_{0}^{4} \frac{1 - \sqrt{u}}{\sqrt{u}} \, du$ 0

31

$$\int_{1}^{32} x^{-6/5} \, dx$$

$$\left. -5x^{-1/5} \right|_{1}^{32}$$

then solve



In Exercises 27–40, evaluate each integral using Part 2 of the Fundamental Theorem. Support your answer with NINT if you are unsure.

27. $\displaystyle\int_{1/2}^{3}\left(2-\frac{1}{x}\right)dx$ $\quad 5-\ln 6\approx 3.208$

28. $\displaystyle\int_{2}^{-1} 3^x\,dx$ $\quad -\dfrac{26}{3\ln 3}\approx -7.889$

29. $\displaystyle\int_{0}^{1}(x^2+\sqrt{x})\,dx$ $\quad 1$

30. $\displaystyle\int_{0}^{5} x^{3/2}\,dx$ $\quad 10\sqrt{5}\approx 22.361$

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33. $\displaystyle\int_{0}^{\pi}\sin x\,dx$ $\quad 2$

34. $\displaystyle\int_{0}^{\pi}(1+\cos x)\,dx$ $\quad \pi$

35. $\displaystyle\int_{0}^{\pi/3} 2\sec^2\theta\,d\theta$ $\quad 2\sqrt{3}$

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37. $\displaystyle\int_{\pi/4}^{3\pi/4}\csc x\cot x\,dx$ $\quad 0$

38. $\displaystyle\int_{0}^{\pi/3} 4\sec x\tan x\,dx$ $\quad 4$

39. $\displaystyle\int_{-1}^{1}(r+1)^2\,dr$ $\quad \dfrac{8}{3}$

40. $\displaystyle\int_{0}^{4}\frac{1-\sqrt{u}}{\sqrt{u}}\,du$ $\quad 0$

30 $\quad \displaystyle\int_0^5 x^{3/2}\,dx$

$$\frac{2}{5}x^{5/2}\Big|_0^5$$

$$\frac{2}{5}(5)^{5/2}\ \cancel{-\frac{2}{5}(0)^{5/2}}$$

$$\frac{2(\sqrt{5})^5}{5}=\frac{2\cdot\cancel{25}^{\,5}\sqrt{5}}{\cancel{5}}$$

In Exercises 27–40, evaluate each integral using Part 2 of the Fundamental Theorem. Support your answer with NINT if you are unsure.

27. $\int_{1/2}^{3} \left(2 - \frac{1}{x}\right) dx$ $5 - \ln 6 \approx 3.208$

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37. $\int_{\pi/4}^{3\pi/4} \csc x \cot x \, dx$ 0

38. $\int_{0}^{\pi/3} \underline{4 \sec x \tan x} \, dx$ 4

39. $\int_{-1}^{1} (r + 1)^2 \, dr$ $\frac{8}{3}$

40. $\int_{0}^{4} \frac{1 - \sqrt{u}}{\sqrt{u}} \, du$ 0

$\sec \cancel{x} \tan x$?

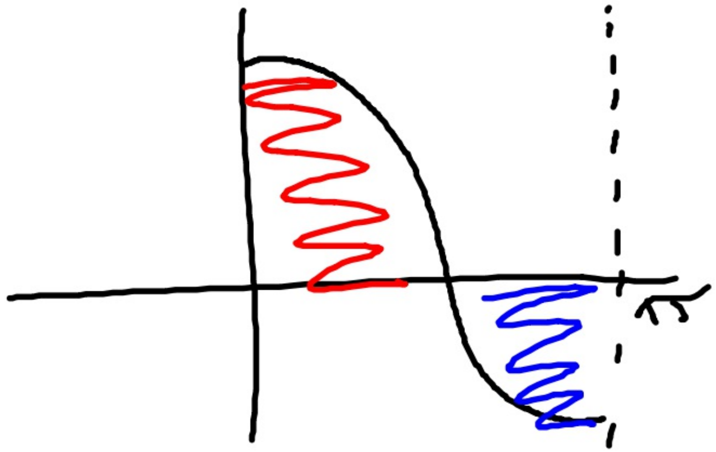
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

38 $4\int_{0}^{\pi/3} \sec x \tan x \, dx$

$$4 \Big|_{0}^{\pi/3} \sec x$$

$$4(2-1) = 4(1) = 4$$

What happens if we integrate cosine from 0 to π?



$+ = 0$

$$\int_0^{\pi} \cos x$$

NO!!

vs.

Q find the area enclosed between the graph and x-axis.

$\neq 0$

EXAMPLE 6 Finding Area Using Antiderivatives

Find the area of the region between the curve $y = 4 - x^2$, $0 \le x \le 3$, and the x-axis.

SOLUTION

The curve crosses the x-axis at $x = 2$, partitioning the interval $[0, 3]$ into two subintervals, on each of which $f(x) = 4 - x^2$ will not change sign.

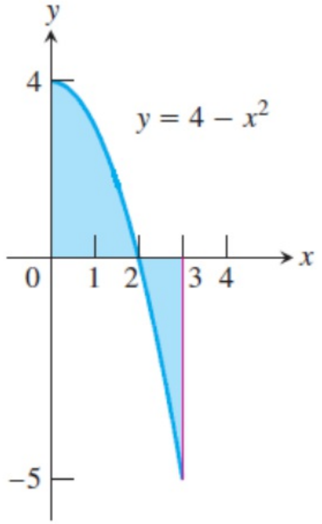
We can see from the graph (Figure 5.28) that $f(x) > 0$ on $[0, 2)$ and $f(x) < 0$ on $(2, 3]$.

Over $[0, 2]$: $\displaystyle\int_0^2 (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_0^2 = \frac{16}{3}.$

Over $[2, 3]$: $\displaystyle\int_2^3 (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_2^3 = -\frac{7}{3}.$

The area of the region is $\left|\frac{16}{3}\right| + \left|-\frac{7}{3}\right| = \frac{23}{3}.$

Now try Exercise 41.



Be careful about this problem...it didn't ask for the integration, it asked for the area within the bounded area!

....IF I asked for simply the integration from 0 to 3, the answer is 3 (you see why...?)

Is there another way to approach these problems…?

EXAMPLE 7 Finding Area Using NINT

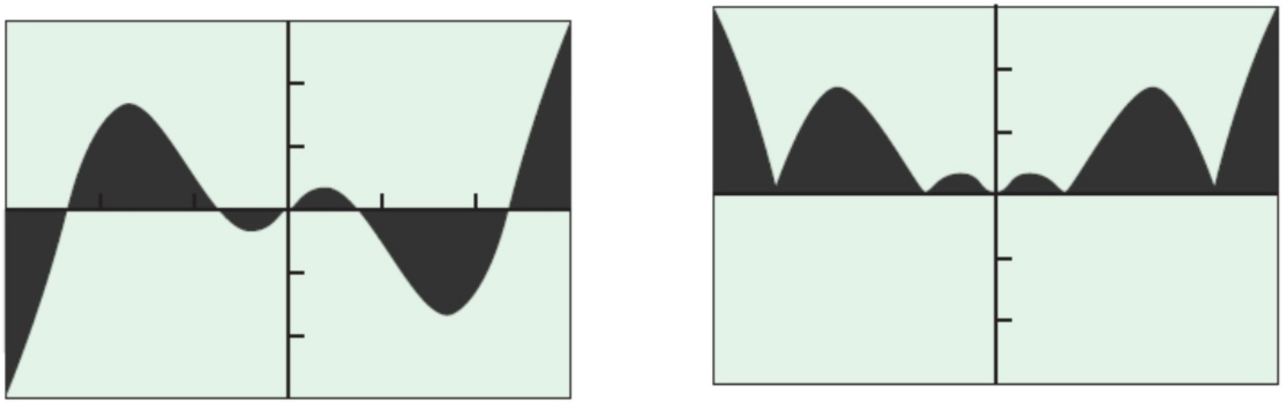
Find the area of the region between the curve $y = x \cos 2x$ and the x-axis over the interval $-3 \le x \le 3$ (Figure 5.29).

SOLUTION

Rounded to two decimal places, we have

$$\text{NINT}\,(|x \cos 2x|, x, -3, 3) = 5.43.$$

Now try Exercise 51.



[–3, 3] by [–3, 3]

(a)

[–3, 3] by [–3, 3]

(b)

These problems aren't assigned, but isn't this cool?!

In Exercises 41–44, find the total area of the region between the curve and the x-axis.

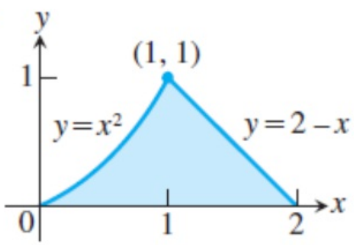
41. $y = 2 - x, \quad 0 \le x \le 3$ $\frac{5}{2}$

42. $y = 3x^2 - 3, \quad -2 \le x \le 2$ 12

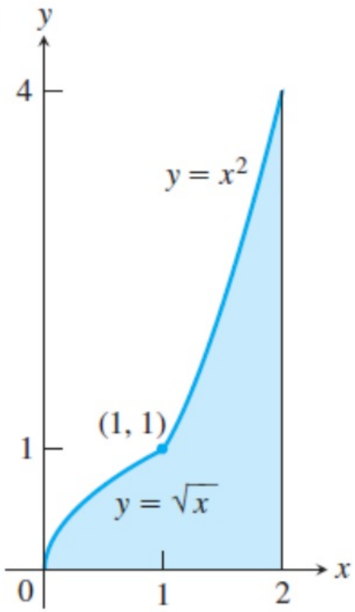
43. $y = x^3 - 3x^2 + 2x, \quad 0 \le x \le 2$ $\frac{1}{2}$

44. $y = x^3 - 4x, \quad -2 \le x \le 2$ 8

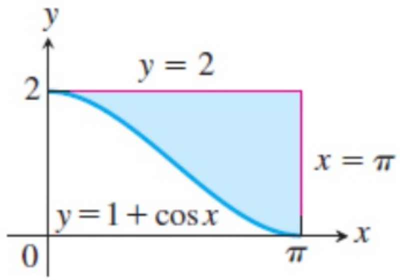
In Exercises 45–48, find the area of the shaded region.

45. 

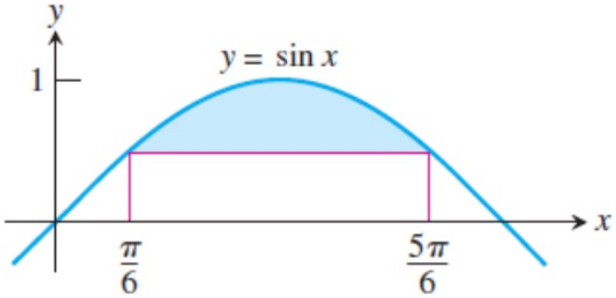
$\frac{5}{6}$

46. 

3

47. 

π

48. 

$\sqrt{3}-\frac{\pi}{3}$