6-1 Operations of Functions

KeyConcept Operations on Functions

Operation	Definition	Example Let $f(x) = 2x$ and $g(x) = -x + 5$.
Addition	$(f + g)(x) = f(x) + g(x)$	$2x + (-x + 5) = x + 5$
Subtraction	$(f - g)(x) = f(x) - g(x)$	$2x - (-x + 5) = 3x - 5$
Multiplication	$(f \cdot g)(x) = f(x) \cdot g(x)$	$2x(-x + 5) = -2x^2 + 10x$
Division	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$	$\frac{2x}{-x + 5}, x \neq 5$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$ Division of functions

$$= \frac{x^2 + 7x + 12}{3x - 4}, x \neq \frac{4}{3}$$ Substitution

Because $x = \frac{4}{3}$ makes the denominator $3x - 4 = 0$, $\frac{4}{3}$ is excluded from the domain of $\left(\frac{f}{g}\right)(x)$.

Examples 1–2 Find $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$. Indicate any restrictions in domain or range. **1, 2. See margin.**

1. $f(x) = x + 2$
   $g(x) = 3x - 1$

2. $f(x) = x^2 - 5$
   $g(x) = -x + 8$

① $(x+2)+(3x-1) = 4x+1$

$(x+2)-(3x-1) = -2x+3$

	$3x$	-1
x	$3x^2$	$-x$
2	$6x$	-2

$3x^2+5x-2$

$$\frac{x+2}{3x-1}$$

$3x-1 \neq 0$

$x \neq 1/3$

1. $(f+g)(x) = 4x + 1$; $(f-g)(x) = -2x + 3$; $(f \cdot g)(x) = 3x^2 + 5x - 2$; $\left(\frac{f}{g}\right)(x) = \frac{x+2}{3x-1}$, $x \neq \frac{1}{3}$

2. $(f+g)(x) = x^2 - x + 3$; $(f-g)(x) = x^2 + x - 13$; $(f \cdot g)(x) = -x^3 + 8x^2 + 5x - 40$; $\left(\frac{f}{g}\right)(x) = \frac{x^2-5}{-x+8}$, $x \neq 8$

2 Composition of Functions

Composition of Functions Another method used to combine functions is a composition of functions. In a **composition of functions**, the results of one function are used to evaluate a second function.

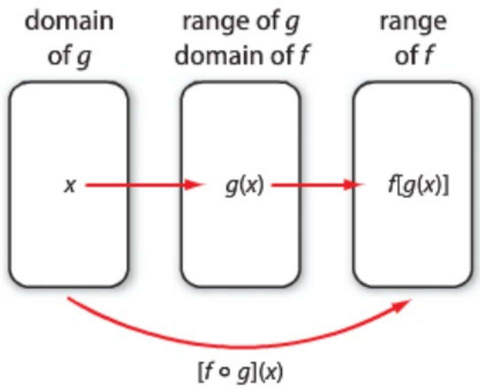
> **ReadingMath**
>
> **Composition of Functions** The composition of f and g, denoted by $f \circ g$ or $f[g(x)]$, is read *f of g*.

KeyConcept Composition of Functions

Words Suppose f and g are functions such that the range of g is a subset of the domain of f. Then the composition function $f \circ g$ can be described by

$$[f \circ g](x) = f[g(x)].$$

Model



The composition of two functions may not exist. Given two functions f and g, $[f \circ g](x)$ is defined only if the range of $g(x)$ is a subset of the domain of f. Likewise, $[g \circ f](x)$ is defined only if the range of $f(x)$ is a subset of the domain of g.

Example 3 Compose Functions

For each pair of functions, find $[f \circ g](x)$ and $[g \circ f](x)$, if they exist. State the domain and range for each composed function.

a. $f = \{(1, 8), (0, 13), (15, 11), (14, 9)\}$, $g = \{(8, 15), (5, 1), (10, 14), (9, 0)\}$

To find $f \circ g$, evaluate $g(x)$ first. Then use the range to evaluate $f(x)$.

$f[g(8)] = f(15)$ or 11 — $g(8) = 15$

$f[g(10)] = f(14)$ or 9 — $g(10) = 14$

$f[g(5)] = f(1)$ or 8 — $g(5) = 1$

$f[g(9)] = f(0)$ or 13 — $g(9) = 0$

$f \circ g = \{(8, 11), (5, 8), (10, 9), (9, 13)\}$, D = {5, 8, 9, 10}, R = {8, 9, 11, 13}

To find $g \circ f$, evaluate $f(x)$ first. Then use the range to evaluate $g(x)$.

$g[f(1)] = g(8)$ or 15 — $f(1) = 8$

$g[f(15)] = g(11)$ — $g(11)$ is undefined.

$g[f(0)] = g(13)$ — $g(13)$ is undefined.

$g[f(14)] = g(9)$ or 0 — $f(14) = 0$

Because 11 and 13 are not in the domain of g, $g \circ f$ is undefined for $x = 11$ and $x = 13$. However, $g[f(1)] = 15$ and $g[f(14)] = 0$, so $g \circ f = \{(1, 15), (14, 0)\}$, D = {5, 8, 9, 10}, and R = {8, 9, 11, 13}.

b. $f(x) = 2a - 5$, $g(x) = 4a$

$[f \circ g](x) = f[g(x)]$	Composition of functions	$[g \circ f](x) = g[f(x)]$
$= f(4a)$	Substitute.	$= g(2a - 5)$
$= 2(4a) - 5$	Substitute again.	$= 4(2a - 5)$
$= 8a - 5$	Simplify.	$= 8a - 20$

For $[f \circ g](x)$, D = {all real numbers} and R = {all real numbers}, and for $[g \circ f](x)$, D = {all real numbers} and R = {all real numbers}.

Example 3 For each pair of functions, find $f \circ g$ and $g \circ f$, if they exist. State the domain and range for each composed function. **3, 4. See margin.**

3. $f = \{(2, 5), (6, 10), (12, 9), (7, 6)\}$
   $g = \{(9, 11), (6, 15), (10, 13), (5, 8)\}$

4. $f = \{(-5, 4), (14, 8), (12, 1), (0, -3)\}$
   $g = \{(-2, -4), (-3, 2), (-1, 4), (5, -6)\}$

Find $[f \circ g](x)$ and $[g \circ f](x)$, if they exist. State the domain and range for each composed function. **5, 6. See margin.**

5. $f(x) = -3x$
   $g(x) = 5x - 6$

6. $f(x) = x + 4$
   $g(x) = x^2 + 3x - 10$

(5) $f(x) = -3x$

$f(g(x)) = -3(5x - 6)$

$g(f(x)) = 5(-3x) - 6$

$f(g(x))$

$g(f(x))$

3. $f \circ g$ is undefined, D = ∅, R = ∅; $g \circ f = \{(2, 8), (6, 13), (12, 11), (7, 15)\}$, D = {2, 6, 7, 12}, R = {8, 11, 13, 15}.

4. $f \circ g$ is undefined, D = ∅, R = ∅; $g \circ f = \{(0, 2)\}$, D = {0}, R = {2}.

5. $[f \circ g](x) = -15x + 18$, R = {all multiples of 3}
$[g \circ f](x) = -15x - 6$

6. $[f \circ g](x) = x^2 + 3x - 6$;
$[g \circ f](x) = x^2 + 11x + 18$

Real-World Example 4 Use Composition of Functions

SHOPPING A new car dealer is discounting all new cars by 12%. At the same time, the manufacturer is offering a $1500 rebate on all new cars. Mr. Navarro is buying a car that is priced $24,500. Will the final price be lower if the discount is applied before the rebate or if the rebate is applied before the discount?

Understand Let x represent the original price of a new car, $d(x)$ represent the price of a car after the discount, and $r(x)$ the price of the car after the rebate.

Plan Write equations for $d(x)$ and $r(x)$.

The original price is discounted by 12%. $d(x) = x - 0.12x$

There is a $1500 rebate on all new cars. $r(x) = x - 1500$

Solve If the discount is applied *before* the rebate, then the final price of Mr. Navarro's new car is represented by $[r \circ d](24{,}500)$.

$$[r \circ d](x) = r[d(x)]$$

$$\begin{aligned}
[r \circ d](24{,}500) &= r[24{,}500 - 0.12(24{,}500)] \\
&= r(24{,}500 - 2940) \\
&= r(21{,}560) \\
&= 21{,}560 - 1500 \\
&= 20{,}060
\end{aligned}$$

If the rebate is given *before* the discount is applied, then the final price of Mr. Navarro's car is represented by $[d \circ r](24{,}500)$.

$$[d \circ r](x) = d[r(x)]$$

$$\begin{aligned}
[d \circ r](24{,}500) &= d(24{,}500 - 1500) \\
&= d(23{,}000) \\
&= 23{,}000 - 0.12(23{,}000) \\
&= 23{,}000 - 2760 \\
&= 20{,}240
\end{aligned}$$

$[r \circ d](24{,}500) = 20{,}060$ and $[d \circ r](24{,}500) = 20{,}240$. So, the final price of the car is less when the discount is applied before the rebate.

Example 4

7. CCSS **MODELING** Dora has 8% of her earnings deducted from her paycheck for a college savings plan. She can choose to take the deduction either before taxes are withheld, which reduces her taxable income, or after taxes are withheld. Dora's tax rate is 17.5%. If her pay before taxes and deductions is $950, will she save more money if the deductions are taken before or after taxes are withheld? Explain. **See margin.**

7. Either way, she will have $228.95 taken from her paycheck. If she takes the college savings plan deduction before taxes, $76 will go to her college plan and $152.95 will go to taxes. If she takes the college savings plan deduction after taxes, only $62.70 will go to her college plan and $166.25 will go to taxes.

Practice and Problem Solving

Extra Practice is o

Examples 1–2 **Find $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$. Indicate any restrictions in domain or range.** **8–15. See margin.**

8. $f(x) = 2x$
   $g(x) = -4x + 5$

9. $f(x) = x - 1$
   $g(x) = 5x - 2$

10. $f(x) = x^2$
    $g(x) = -x + 1$

11. $f(x) = 3x$
    $g(x) = -2x + 6$

12. $f(x) = x - 2$
    $g(x) = 2x - 7$

13. $f(x) = x^2$
    $g(x) = x - 5$

14. $f(x) = -x^2 + 6$
    $g(x) = 2x^2 + 3x - 5$

15. $f(x) = 3x^2 - 4$
    $g(x) = x^2 - 8x + 4$

16. **POPULATION** In a particular county, the population of the two largest cities can be modeled by $f(x) = 200x + 25$ and $g(x) = 175x - 15$, where x is the number of years since 2000 and the population is in thousands.
    a. What is the population of the two cities combined after any number of years? $(f+g)(x) = 375x + 10$
    b. What is the difference in the populations of the two cities? $(f-g)(x) = 25x + 40$

Example 3 **For each pair of functions, find $f \circ g$ and $g \circ f$, if they exist. State the domain and range for each composed function.** **17–20. See Chapter 6 Answer Appendix.**

17. $f = \{(-8, -4), (0, 4), (2, 6), (-6, -2)\}$
    $g = \{(4, -4), (-2, -1), (-4, 0), (6, -5)\}$

18. $f = \{(-7, 0), (4, 5), (8, 12), (-3, 6)\}$
    $g = \{(6, 8), (-12, -5), (0, 5), (5, 1)\}$

19. $f = \{(5, 13), (-4, -2), (-8, -11), (3, 1)\}$
    $g = \{(-8, 2), (-4, 1), (3, -3), (5, 7)\}$

20. $f = \{(-4, -14), (0, -6), (-6, -18), (2, -2)\}$
    $g = \{(-6, 1), (-18, 13), (-14, 9), (-2, -3)\}$

Additional Answers

8. $(f+g)(x) = -2x + 5$; $(f-g)(x) = 6x - 5$; $(f \cdot g)(x) = -8x^2 + 10x$; $\left(\frac{f}{g}\right)(x) = \frac{2x}{-4x+5}, x \neq \frac{5}{4}$

9. $(f+g)(x) = 6x - 3$; $(f-g)(x) = -4x + 1$; $(f \cdot g)(x) = 5x^2 - 7x + 2$; $\left(\frac{f}{g}\right)(x) = \frac{x-1}{5x-2}, x \neq \frac{2}{5}$

10. $(f+g)(x) = x^2 - x + 1$; $(f-g)(x) = x^2 + x - 1$; $(f \cdot g)(x) = -x^3 + x^2$; $\left(\frac{f}{g}\right)(x) = \frac{x^2}{-x+1}, x \neq 1$

11. $(f+g)(x) = x + 6$; $(f-g)(x) = 5x - 6$; $(f \cdot g)(x) = -6x^2 + 18x$; $\left(\frac{f}{g}\right)(x) = \frac{3x}{-2x+6}, x \neq 3$

12. $(f+g)(x) = 3x - 9$; $(f-g)(x) = -x + 5$;
    $(f \cdot g)(x) = 2x^2 - 11x + 14$;
    $\left(\frac{f}{g}\right)(x) = \frac{x-2}{2x-7}, x \neq \frac{7}{2}$

13. $(f+g)(x) = x^2 + x - 5$;
    $(f-g)(x) = x^2 - x + 5$;
    $(f \cdot g)(x) = x^3 - 5x^2$;
    $\left(\frac{f}{g}\right)(x) = \frac{x^2}{x-5}, x \neq 5$

14. $(f+g)(x) = x^2 + 3x + 1$;
    $(f-g)(x) = -3x^2 - 3x + 11$;
    $(f \cdot g)(x) = -2x^4 - 3x^3 + 17x^2 + 18x - 30$;
    $\left(\frac{f}{g}\right)(x) = \frac{-x^2+6}{2x^2+3x-5}, x \neq 1$ or $-\frac{5}{2}$

15. $(f+g)(x) = 4x^2 - 8x$; $(f-g)(x) = 2x^2 + 8x - 8$;
    $(f \cdot g)(x) = 3x^4 - 24x^3 + 8x^2 + 32x - 16$; $\left(\frac{f}{g}\right)(x) = \frac{3x^2-4}{x^2-8x+4}, x \neq 4 \pm 2\sqrt{3}$