6-2 Inverse Function and Relations

KeyConcept Inverse Relations

Words	Two relations are inverse relations if and only if whenever one relation contains the element (a, b), the other relation contains the element (b, a).
Example	A and B are inverse relations.
	$A = \{(1, 5), (2, 6), (3, 7)\}$ $B = \{(5, 1), (6, 2), (7, 3)\}$

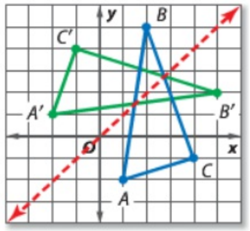
PT

Example 1 Find an Inverse Relation

GEOMETRY **The vertices of $\triangle ABC$ can be represented by the relation $\{(1, -2), (2, 5), (4, -1)\}$. Find the inverse of this relation. Describe the graph of the inverse.**

Graph the relation. To find the inverse, exchange the coordinates of the ordered pairs. The inverse of the relation is $\{(-2, 1), (5, 2), (-1, 4)\}$.

Plotting these points shows that the ordered pairs describe the vertices of $\triangle A'B'C'$ as a reflection of $\triangle ABC$ in the line $y = x$.



GuidedPractice $\{(-3, -8), (-6, -8), (-6, -3)\}$; It is a reflection in the line $y = x$.

1. **GEOMETRY** The ordered pairs of the relation $\{(-8, -3), (-8, -6), (-3, -6)\}$ are the coordinates of the vertices of a right triangle. Find the inverse of this relation. Describe the graph of the inverse.

As with relations, the ordered pairs of **inverse functions** are also related. We can write the inverse of the function $f(x)$ as $f^{-1}(x)$.

Example 1 **Find the inverse of each relation.**

1. $\{(-9, 10), (1, -3), (8, -5)\}$

2. $\{(-2, 9), (4, -1), (-7, 9), (7, 0)\}$

nple Let $f(x) = x - 4$ and represent its inverse as $f^{-1}(x) = x + 4$.

Evaluate $f(6)$.

$f(x) = x - 4$

$f(6) = 6 - 4$ or 2

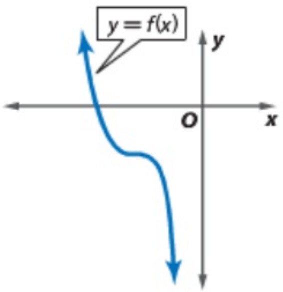
Evaluate $f^{-1}(2)$.

$f^{-1}(x) = x + 4$

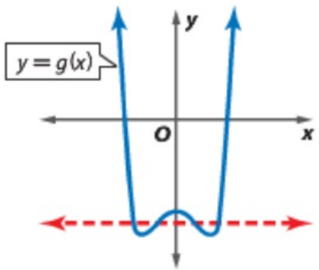
$f^{-1}(2) = 2 + 4$ or 6

Because $f(x)$ and $f^{-1}(x)$ are inverses, $f(6) = 2$ and $f^{-1}(2) = 6$.

When the inverse of a function is a function, the original function is one-to-one. Recall that the vertical line test can be used to determine whether a relation is a function. Similarly, the *horizontal line test* can be used to determine whether the inverse of a function is also a function.



No horizontal line can be drawn so that it passes through more than one point. The inverse of $y = f(x)$ is a function.



A horizontal line can be drawn that passes through more than one point. The inverse of $y = g(x)$ is not a function.

The inverse of a function can be found by exchanging the domain and the range.

Example 2 Find and Graph an Inverse

Find the inverse of each function. Then graph the function and its inverse.

a. $f(x) = 2x - 5$

Step 1 Rewrite the function as an equation relating x and y.

$f(x) = 2x - 5 \rightarrow y = 2x - 5$

Step 2 Exchange x and y in the equation. $x = 2y - 5$

Step 3 Solve the equation for y.

$x = 2y - 5$ — Inverse of $y = 2x - 5$

$x + 5 = 2y$ — Add 5 to each side.

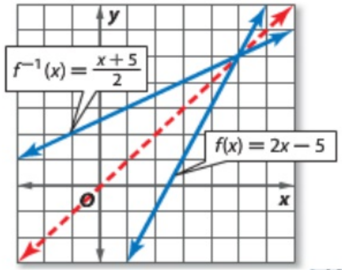
$\frac{x+5}{2} = y$ — Divide each side by 2.

Step 4 Replace y with $f^{-1}(x)$.

$y = \frac{x+5}{2} \rightarrow f^{-1}(x) = \frac{x+5}{2}$

The inverse of $f(x) = 2x - 5$ is $f^{-1}(x) = \frac{x+5}{2}$.

The graph of $f^{-1}(x) = \frac{x+5}{2}$ is the reflection of the graph of $f(x) = 2x - 5$ in the line $y = x$.



b. $f(x) = x^2 + 1$

Step 1 $f(x) = x^2 + 1 \rightarrow y = x^2 + 1$

Step 2 $x = y^2 + 1$

Step 3

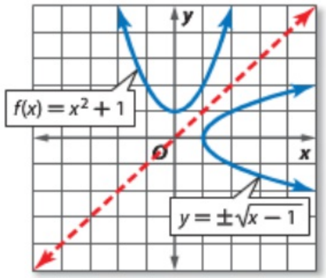
$x = y^2 + 1$

$x - 1 = y^2$

$\pm\sqrt{x-1} = y$ — Take the square root of each side.

Step 4 $y = \pm\sqrt{x-1}$

Graph $y = \pm\sqrt{x-1}$ by reflecting the graph of $f(x) = x^2 + 1$ in the line $y = x$.

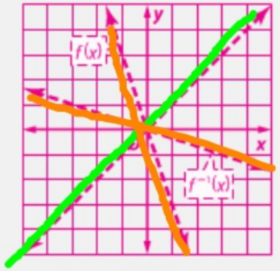


Example 2 Find the inverse of each function. Then graph the function and its inverse.

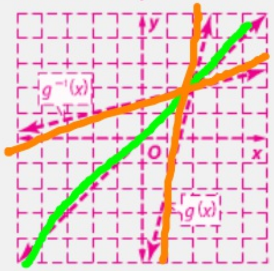
3. $f(x) = -3x$ 4. $g(x) = 4x - 6$ 5. $h(x) = x^2 - 3$

Additional Answers

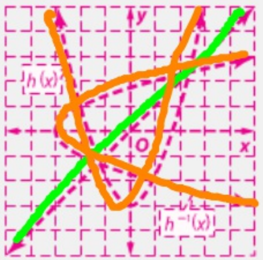
3. $f^{-1}(x) = -\frac{1}{3}x$



4. $g^{-1}(x) = \frac{x + 6}{4}$



5. $h^{-1}(x) = \pm\sqrt{x + 3}$



(3) $y = -3x$

$x = -3y$

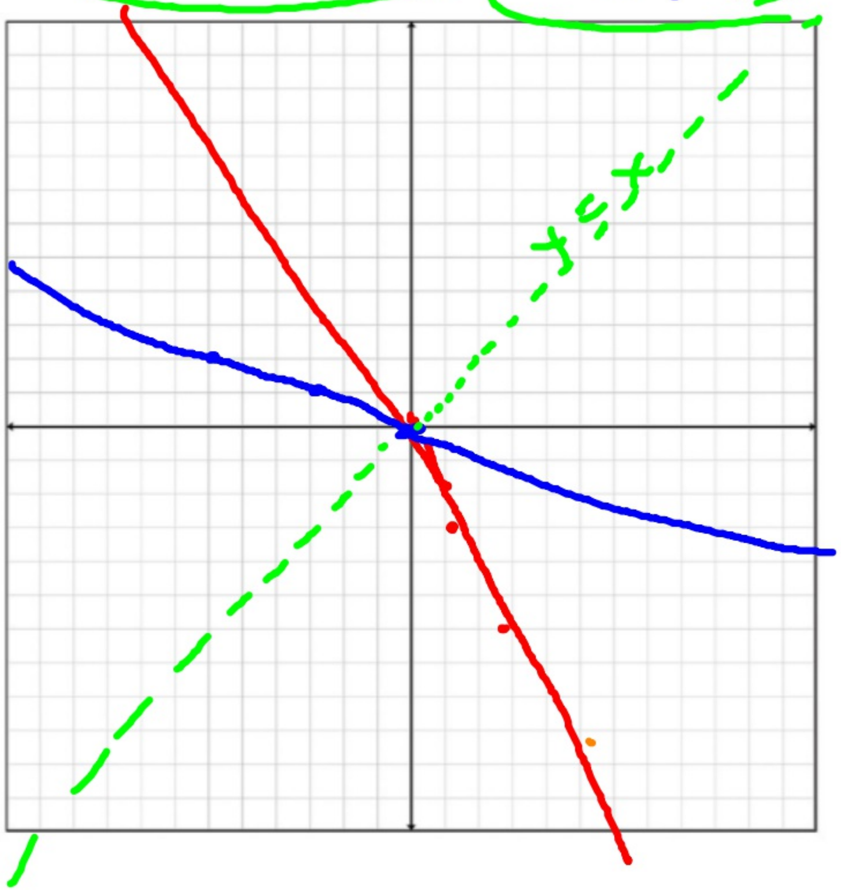
$\frac{x}{-3} = \frac{-3y}{-3}$

$y = -\frac{1}{3}x$

$f^{-1}(x) = -\frac{1}{3}x$

x	f(x)
0	0
1	-3
2	-6

x	$f^{-1}(x)$
0	0
-3	1
-6	2



KeyConcept Inverse Functions

Words	Two functions f and g are inverse functions if and only if both of their compositions are the identity function.
Symbols	$f(x)$ and $g(x)$ are inverses if and only if $[f \circ g](x) = x$ and $[g \circ f](x) = x$.

Example 3 Verify that Two Functions are Inverses

Determine whether each pair of functions are inverse functions. Explain your reasoning.

a. $f(x) = 3x + 9$ **and** $g(x) = \frac{1}{3}x - 3$

Verify that the compositions of $f(x)$ and $g(x)$ are identity functions.

$$[f \circ g](x) = f[g(x)] = f\left(\frac{1}{3}x - 3\right) = 3\left(\frac{1}{3}x - 3\right) + 9 = x - 9 + 9 \text{ or } x$$

$$[g \circ f](x) = g[f(x)] = g(3x + 9) = \frac{1}{3}(3x + 9) - 3 = x + 3 - 3 \text{ or } x$$

The functions are inverses because $[f \circ g](x) = [g \circ f](x) = x$.

b. $f(x) = 4x^2$ **and** $g(x) = 2\sqrt{x}$

$$[f \circ g](x) = f\left(2\sqrt{x}\right) = 4\left(2\sqrt{x}\right)^2 = 4(4x) \text{ or } 16x$$

Because $[f \circ g](x) \neq x$, $f(x)$ and $g(x)$ are not inverses.

Example 3 Determine whether each pair of functions are inverse functions. Write *yes* or *no*.

6. $f(x) = x - 7$
$g(x) = x + 7$ yes

7. $f(x) = \frac{1}{2}x + \frac{3}{4}$
$g(x) = 2x - \frac{4}{3}$ no

8. $f(x) = 2x^3$
$g(x) = \frac{1}{3}\sqrt{x}$ no

7

$(g \circ f)(x) = g(f(x)) = 2\left(\frac{1}{2}x + \frac{3}{4}\right)$

$= x + \frac{6}{4} - \frac{4}{3}$

$\neq x$

nope

Practice and Problem Solving

Extra Practice is on page R6.

Example 1 Find the inverse of each relation. **9. {(6, −8), (−2, 6), (−3, 7)}** **10. {(7, 7), (9, 4), (−7, 3)}**

11. {(−1, 8), (−1, −8), (−8, −2), (8, 2)} **12. {(3, 4), (−4, −4), (−5, −3), (2, 5)}**

9. $\{(-8, 6), (6, -2), (7, -3)\}$ **10.** $\{(7, 7), (4, 9), (3, -7)\}$

11. $\{(8, -1), (-8, -1), (-2, -8), (2, 8)\}$ **12.** $\{(4, 3), (-4, -4), (-3, -5), (5, 2)\}$

13. $\{(1, -5), (2, 6), (3, -7), (4, 8), (5, -9)\}$ **{(−5, 1), (6, 2), (−7, 3), (8, 4), (−9, 5)}**

14. $\{(3, 0), (5, 4), (7, -8), (9, 12), (11, 16)\}$ **{(0, 3), (4, 5), (−8, 7), (12, 9), (16, 11)}**

Example 2 CCSS **SENSE-MAKING** Find the inverse of each function. Then graph the function and its inverse. **15–26. See Chapter 6 Answer Appendix.**

15. $f(x) = x + 2$ **16.** $g(x) = 5x$ **17.** $f(x) = -2x + 1$

18. $h(x) = \frac{x-4}{3}$ **19.** $f(x) = -\frac{5}{3}x - 8$ **20.** $g(x) = x + 4$

21. $f(x) = 4x$ **22.** $f(x) = -8x + 9$ **23.** $f(x) = 5x^2$

24. $h(x) = x^2 + 4$ **25.** $f(x) = \frac{1}{2}x^2 - 1$ **26.** $f(x) = (x + 1)^2 + 3$

Example 3 Determine whether each pair of functions are inverse functions. Write *yes* or *no*.

27. $f(x) = 2x + 3$, $g(x) = 2x - 3$ **no**

28. $f(x) = 4x + 6$, $g(x) = \frac{x-6}{4}$ **yes**

29. $f(x) = -\frac{1}{3}x + 3$, $g(x) = -3x + 9$ **yes**

30. $f(x) = -6x$, $g(x) = \frac{1}{6}x$ **no**

31. $f(x) = \frac{1}{2}x + 5$, $g(x) = 2x - 10$ **yes**

32. $f(x) = \frac{x+10}{8}$, $g(x) = 8x - 10$ **yes**

33. $f(x) = 4x^2$, $g(x) = \frac{1}{2}\sqrt{x}$ **yes**

34. $f(x) = \frac{1}{3}x^2 + 1$, $g(x) = \sqrt{3x - 3}$ **yes**

35. $f(x) = x^2 - 9$, $g(x) = x + 3$ **no**

36. $f(x) = \frac{2}{3}x^3$, $g(x) = \sqrt{\frac{2}{3}x}$ **no**

37. $f(x) = (x + 6)^2$, $g(x) = \sqrt{x} - 6$ **yes**

38. $f(x) = 2\sqrt{x - 5}$, $g(x) = \frac{1}{4}x^2 - 5$ **no**

29

$g(f(x))$

$= -3(-\frac{1}{3}x + 3) + 9$

$= x - 9 + 9 = x$ ✓

$f(g(x)) = -\frac{1}{3}(-3x + 9) + 3$

$= x - 3 + 3 = x$ ✓

Lesson 6-1

17. $f \circ g = \{(-4, 4)\}$, D = {−4}, R = {4}; $g \circ f = \{(-8, 0), (0, -4), (2, -5), (-6, -1)\}$, D = {−6, 0, 2}, R = {−5, −4, −1, 0}

18. $f \circ g = \{(6, 12)\}$, D = {6}, R = {12}; $g \circ f = \{(-7, 5), (4, 1), (-3, 8)\}$, D = {−7, −3, 4}, R = {1, 5, 8}

19. $f \circ g$ is undefined, D = ∅, R = ∅; $g \circ f$ is undefined, D = ∅, R = ∅.

20. $f \circ g$ is undefined, D = ∅, R = ∅; $g \circ f = \{(-4, 9), (0, 1), (-6, 13), (2, -3)\}$, D = {−6, −4, 0, 2}, R = {−3, 1, 9, 13}.

28–35. Domains and ranges are all real numbers unless otherwise specified.

27. $[f \circ g](x) = 2x + 10$, D = {all real numbers}, R = {all even numbers}; $[g \circ f](x) = 2x + 5$, D = {all real numbers}, R = {all odd numbers}

28. $[f \circ g](x) = 3x - 24$; $[g \circ f](x) = 3x + 8$

29. $[f \circ g](x) = 3x - 2$; $[g \circ f](x) = 3x + 8$

30. $[f \circ g](x) = x^2 - 14$, $R = \{y \mid y \geq -14\}$; $[g \circ f](x) = x^2 - 8x + 6$, $R = \{y \mid y \geq -10\}$

31. $[f \circ g](x) = x^2 - 6x - 2$, $R = \{y \mid y \geq -11\}$; $[g \circ f](x) = x^2 + 6x - 8$, $R = \{y \mid y \geq -17\}$

32. $[f \circ g](x) = 32x^2 + 44x + 16$, $R = \{y \mid y \geq 0.875\}$; $[g \circ f](x) = 8x^2 - 4x + 7$, $R = \{y \mid y \geq 6.5\}$

33. $[f \circ g](x) = 4x^3 + 7$; $[g \circ f](x) = 64x^3 - 48x^2 + 12x + 1$

34. $[f \circ g](x) = x^4 + 3x^2 + 1$, $R = \{y \mid y \geq 1\}$; $[g \circ f](x) = x^4 + 6x^3 + 11x^2 + 6x + 1$, $R = \{y \mid y \geq 0\}$

35. $[f \circ g](x) = 128x^4 + 96x^3 + 18x^2$, $R = \{y \mid y \geq 0\}$; $[g \circ f](x) = 32x^4 + 6x^2$, $R = \{y \mid y \geq 0\}$