6-5 Operations with Radical Expressions

KeyConcept Product Property of Radicals

Words For any real numbers a and b and any integer $n > 1$, $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$, if n is even and a and b are both nonnegative or if n is odd.

Examples $\sqrt{2} \cdot \sqrt{8} = \sqrt{16}$ or 4 and $\sqrt[3]{3} \cdot \sqrt[3]{9} = \sqrt[3]{27}$ or 3

Example 1 Simplify Expressions with the Product Property

Simplify.

a. $\sqrt{32x^8}$

$$\sqrt{32x^8} = \sqrt{4^2 \cdot 2 \cdot (x^4)^2} \quad \text{Factor into squares.}$$

$$= \sqrt{4^2} \cdot \sqrt{(x^4)^2} \cdot \sqrt{2} \quad \text{Product Property of Radicals}$$

$$= 4x^4\sqrt{2} \quad \text{Simplify.}$$

b. $\sqrt[4]{16a^{24}b^{13}}$

$$\sqrt[4]{16a^{24}b^{13}} = \sqrt[4]{2^4 \cdot (a^6)^4(b^3)^4 \cdot b} \quad \text{Factor into squares.}$$

$$= \sqrt[4]{2^4} \cdot \sqrt[4]{(a^6)^4} \cdot \sqrt[4]{(b^3)^4} \cdot \sqrt[4]{b} \quad \text{Product Property of Radicals}$$

$$= 2a^6|b^3|\sqrt[4]{b} \quad \text{Simplify.}$$

In this case, the absolute value symbols are not necessary because in order for $\sqrt[4]{16a^{24}b^{13}}$ to be defined, b must be nonnegative.

Thus, $\sqrt[4]{16a^{24}b^{13}} = 2a^6b^3\sqrt[4]{b}$.

KeyConcept Product Property of Radicals

Words	For any real numbers a and b and any integer $n > 1$, $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$, if n is even and a and b are both nonnegative or if n is odd.
Examples	$\sqrt{2} \cdot \sqrt{8} = \sqrt{16}$ or 4 and $\sqrt[3]{3} \cdot \sqrt[3]{9} = \sqrt[3]{27}$ or 3

Examples 1–5 CCSS **PRECISION** Simplify.

$c \cdot c \cdot c \cdot c \cdot c$

1. $\sqrt{36ab^4c^5}$

$6b^2c^2\sqrt{ac}$

$y \cdot y \cdot y \cdot y \cdot y$

2. $\sqrt{144x^7y^5}$

$12x^3y^2\sqrt{xy}$

$x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$

To eliminate radicals from a denominator or fractions from a radicand, you can use a process called **rationalizing the denominator**. To rationalize a denominator, multiply the numerator and denominator by a quantity so that the radicand has an exact root.

If the denominator is:	Multiply the numerator and denominator by:	Examples
$\sqrt{b}$	$\sqrt{b}$	$\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$ or $\frac{2\sqrt{3}}{3}$
$\sqrt[n]{b^x}$	$\sqrt[n]{b^{n-x}}$	$\frac{5}{\sqrt[3]{2}} = \frac{5}{\sqrt[3]{2}} \cdot \frac{\sqrt[3]{2^2}}{\sqrt[3]{2^2}}$ or $\frac{5\sqrt[3]{4}}{2}$

Example 2 Simplify Expressions with the Quotient Property

Simplify.

a. $\sqrt{\frac{x^6}{y^7}}$

$$\sqrt{\frac{x^6}{y^7}} = \frac{\sqrt{x^6}}{\sqrt{y^7}}$$ Quotient Property

$$= \frac{\sqrt{(x^3)^2}}{\sqrt{(y^3)^2 \cdot y}}$$ Factor into squares.

$$= \frac{\sqrt{(x^3)^2}}{\sqrt{(y^3)^2} \cdot \sqrt{y}}$$ Product Property

$$= \frac{|x^3|}{y^3\sqrt{y}}$$ Simplify.

$$= \frac{|x^3|}{y^3\sqrt{y}} \cdot \frac{\sqrt{y}}{\sqrt{y}}$$ Rationalize the denominator.

$$= \frac{|x^3|\sqrt{y}}{y^4}$$ $\sqrt{y} \cdot \sqrt{y} = y$

b. $\sqrt[4]{\frac{6}{5x}}$

$$\sqrt[4]{\frac{6}{5x}} = \frac{\sqrt[4]{6}}{\sqrt[4]{5x}}$$ Quotient Property

$$= \frac{\sqrt[4]{6}}{\sqrt[4]{5x}} \cdot \frac{\sqrt[4]{5^3x^3}}{\sqrt[4]{5^3x^3}}$$ Rationalize the denominator.

$$= \frac{\sqrt[4]{6 \cdot 5^3x^3}}{\sqrt[4]{5x \cdot 5^3x^3}}$$ Product Property

$$= \frac{\sqrt[4]{750x^3}}{\sqrt[4]{5^4x^4}}$$ Multiply.

$$= \frac{\sqrt[4]{750x^3}}{5x}$$ $\sqrt[4]{5^4x^4} = 5x$

To eliminate radicals from a denominator or fractions from a radicand, you can use a process called **rationalizing the denominator**. To rationalize a denominator, multiply the numerator and denominator by a quantity so that the radicand has an exact root.

If the denominator is:	Multiply the numerator and denominator by:	Examples
$\sqrt{b}$	$\sqrt{b}$	$\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$ or $\frac{2\sqrt{3}}{3}$
$\sqrt[n]{b^x}$	$\sqrt[n]{b^{n-x}}$	$\frac{5}{\sqrt[3]{2}} = \frac{5}{\sqrt[3]{2}} \cdot \frac{\sqrt[3]{2^2}}{\sqrt[3]{2^2}}$ or $\frac{5\sqrt[3]{4}}{2}$

3. $\frac{\sqrt{c^5}}{\sqrt{d^9}}$ $\frac{c^2\sqrt{cd}}{d^5}$

$\frac{c^2\sqrt{c}\sqrt{d}}{d^4\sqrt{d}\sqrt{d}}$

$\sqrt{d^2}$

4. $\sqrt[4]{\frac{5x}{8y}}$ $\frac{\sqrt[4]{10xy^3}}{2y}$

$\sqrt[4]{2y^3}$

$2 \cdot 2 \cdot 2y$

$\frac{\sqrt[4]{5x}}{\sqrt[4]{8y}} \cdot \frac{\sqrt[4]{2y^3}}{\sqrt[4]{2y^3}} = \frac{\sqrt[4]{10xy^3}}{\sqrt[4]{16y^4}}$

$= \frac{\sqrt[4]{10xy^3}}{2y}$

Example 3 Multiply Radicals

Simplify $5\sqrt[3]{-12ab^4} \cdot 3\sqrt[3]{18a^2b^2}$.

$$\begin{aligned}
5\sqrt[3]{-12ab^4} \cdot 3\sqrt[3]{18a^2b^2} &= 5 \cdot 3 \cdot \sqrt[3]{-12ab^4 \cdot 18a^2b^2} && \text{Product Property of Radicals}\\
&= 15 \cdot \sqrt[3]{-2^2 \cdot 3 \cdot ab^4 \cdot 2 \cdot 3^2 \cdot a^2b^2} && \text{Factor constants.}\\
&= 15 \cdot \sqrt[3]{-2^3 \cdot 3^3 \cdot a^3b^6} && \text{Group into cubes if possible.}\\
&= 15 \cdot \sqrt[3]{-2^3} \cdot \sqrt[3]{3^3} \cdot \sqrt[3]{a^3} \cdot \sqrt[3]{b^6} && \text{Product Property of Radicals}\\
&= 15 \cdot (-2) \cdot 3 \cdot a \cdot b^2 && \text{Simplify.}\\
&= -90ab^2 && \text{Multiply.}
\end{aligned}$$

5 $15\sqrt{16x^2} = 15(4x)$

7 $6\sqrt[3]{216x^3y^3} = 6(6xy)$

5. $5\sqrt{2x} \cdot 3\sqrt{8x}$ **60x**

6. $4\sqrt{5a^5} \cdot \sqrt{125a^3}$ **$100a^4$**

7. $3\sqrt[3]{36xy} \cdot 2\sqrt[3]{6x^2y^2}$ **36xy**

8. $\sqrt[4]{3x^3y^2} \cdot \sqrt[4]{27xy^2}$ **$3x|y|$**

ConceptSummary Simplifying Radical Expressions

A radical expression is in simplified form when the following conditions are met.

- The index *n* is as small as possible.
- The radicand contains no factors (other than 1) that are *n*th powers of an integer or polynomial.
- The radicand contains no fractions.
- No radicals appear in a denominator.

Radicals can be added and subtracted in the same manner as monomials. In order to add or subtract, the radicals must be like terms. Radicals are **like radical expressions** if *both* the index and the radicand are identical.

Like: $\sqrt{3b}$ and $4\sqrt{3b}$ | Unlike: $\sqrt{3b}$ and $\sqrt[3]{3b}$ | Unlike: $\sqrt{2b}$ and $\sqrt{3b}$

Example 4 Add and Subtract Radicals

Simplify $\sqrt{98} - 2\sqrt{32}$.

$\sqrt{98} - 2\sqrt{32} = \sqrt{2 \cdot 7^2} - 2\sqrt{4^2 \cdot 2}$ — Factor using squares.

$= \sqrt{7^2} \cdot \sqrt{2} - 2 \cdot \sqrt{4^2} \cdot \sqrt{2}$ — Product Property

$= 7\sqrt{2} - 2 \cdot 4 \cdot \sqrt{2}$ — Simplify radicals.

$= 7\sqrt{2} - 8\sqrt{2}$ — Multiply.

$= -\sqrt{2}$ — $(7 - 8)\sqrt{2} = (-1)(\sqrt{2})$

9. $5\sqrt{32} + \sqrt{27} + 2\sqrt{75}$ $20\sqrt{2} + 13\sqrt{3}$

$8 \cdot 4 \cdot 2$ $\quad 9 \cdot 3$ $\quad 25 \cdot 3$

$4 \cdot 5\sqrt{2} + 3\sqrt{3} + 2 \cdot 5\sqrt{3}$

$= 20\sqrt{2} + 3\sqrt{3} + 10\sqrt{3}$

10. $4\sqrt{40} + 3\sqrt{28} - \sqrt{200}$

Example 5 Multiply Radicals

Simplify $(4\sqrt{3} + 5\sqrt{2})(3\sqrt{2} - 6)$.

$$(4\sqrt{3} + 5\sqrt{2})(3\sqrt{2} - 6) = \overset{F}{4\sqrt{3} \cdot 3\sqrt{2}} + \overset{O}{4\sqrt{3} \cdot (-6)} + \overset{I}{5\sqrt{2} \cdot 3\sqrt{2}} + \overset{L}{5\sqrt{2} \cdot (-6)}$$

$$= 12\sqrt{3 \cdot 2} - 24\sqrt{3} + 15\sqrt{2^2} - 30\sqrt{2}$$ Product Property

$$= 12\sqrt{6} - 24\sqrt{3} + 30 - 30\sqrt{2}$$ Simplify.

11. $(4 + 2\sqrt{5})(3\sqrt{3} + 4\sqrt{5})$ $12\sqrt{3} + 16\sqrt{5} + 40 + 6\sqrt{15}$

	4	$2\sqrt{5}$
$3\sqrt{3}$	$12\sqrt{3}$	$6\sqrt{15}$
$4\sqrt{5}$	$16\sqrt{5}$	40

$(a - b)(a + b)$

12. $(8\sqrt{3} - 2\sqrt{2})(8\sqrt{3} + 2\sqrt{2})$ 184

$64\sqrt{9} - 4\sqrt{4}$

$a^2 - b^2$

$64 \cdot 3 - 4 \cdot 2$

Real-World Example 6 Use a Conjugate to Rationalize a Denominator

ARCHITECTURE Refer to the beginning of the lesson. Use a conjugate to rationalize the denominator and simplify $\frac{2}{\sqrt{5}-1}$.

$$\frac{5}{(\sqrt{2}+3)}\cdot\frac{(\sqrt{2}-3)}{(\sqrt{2}-3)} = \frac{5\sqrt{2}-15}{(\sqrt{2})^2-3^2}$$

$$(a+b)(a-b) \qquad a^2-b^2$$

13. $\frac{5}{\sqrt{2}+3}$ $\frac{15-5\sqrt{2}}{7}$

$$\frac{5\sqrt{2}-15}{-7} = \frac{15-5\sqrt{2}}{7}$$

15. $\frac{4+\sqrt{2}}{\sqrt{2}-3}$ $-2-\sqrt{2}$

15

$$\frac{(4+\sqrt{2})}{(\sqrt{2}-3)}\cdot\frac{(\sqrt{2}+3)}{(\sqrt{2}+3)} \qquad = (\sqrt{2})^2-3^2$$

$$= 4\sqrt{2}+12+2+3\sqrt{2}$$

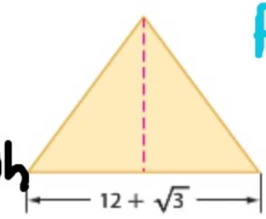
$$= \frac{7\sqrt{2}+14}{2-9} = \frac{7\sqrt{2}+14}{-7}$$

Example 6 **17.** GEOMETRY Find the altitude of the triangle if the area is $189 + 4\sqrt{3}$ square centimeters. $32 - 2\sqrt{3}$ cm

A

$A = \frac{1}{2} b h$

$189 + 4\sqrt{3} = \frac{1}{2}(12 + \sqrt{3})h$



18. $6a^4b^2\sqrt{2b}$
19. $3a^7b\sqrt{ab}$
20. $2a^8b^4\sqrt{6c}$
21. $3|a^3|bc^2\sqrt{2bc}$

Practice and Problem Solving

Extra Practice is on page R6.

Examples 1–4 Simplify. **27.** $32a^5b^3\sqrt{b}$ **29.** $25x^6y^3\sqrt{2xy}$ **30.** $9\sqrt{10} + 8\sqrt{5} + 9\sqrt{2}$

18. $\sqrt{72a^8b^5}$ **19.** $\sqrt{9a^{15}b^3}$ **20.** $\sqrt{24a^{16}b^8c}$ **21.** $\sqrt{18a^6b^3c^5}$

22. $\dfrac{\sqrt{5a^5}}{\sqrt{b^{13}}}$ $\dfrac{a^2\sqrt{5ab}}{b^7}$ **23.** $\sqrt{\dfrac{7x}{10y^3}}$ $\dfrac{\sqrt{70xy}}{10y^2}$ **24.** $\dfrac{\sqrt[3]{6x^2}}{\sqrt[3]{5y}}$ $\dfrac{\sqrt[3]{150x^2y^2}}{5y}$ **25.** $\sqrt[4]{\dfrac{7x^3}{4b^2}}$ $\dfrac{\sqrt[4]{28b^2x^3}}{2|b|}$

26. $3\sqrt{5y} \cdot 8\sqrt{10yz}$ $120y\sqrt{2z}$ **27.** $2\sqrt{32a^3b^5} \cdot \sqrt{8a^7b^2}$ **28.** $6\sqrt{3ab} \cdot 4\sqrt{24ab^3}$ $144ab^2\sqrt{2}$

29. $5\sqrt{x^8y^3} \cdot 5\sqrt{2x^5y^4}$ **30.** $3\sqrt{90} + 4\sqrt{20} + \sqrt{162}$ **31.** $9\sqrt{12} + 5\sqrt{32} - \sqrt{72}$ $18\sqrt{3} + 14\sqrt{2}$

32. $4\sqrt{28} - 8\sqrt{810} + \sqrt{44}$ $8\sqrt{7} - 72\sqrt{10} + 2\sqrt{11}$ **33.** $3\sqrt{54} + 6\sqrt{288} - \sqrt{147}$ $9\sqrt{6} + 72\sqrt{2} - 7\sqrt{3}$

34. GEOMETRY Find the perimeter of the rectangle. $16 + 2\sqrt{3} + 2\sqrt{6}$ ft

35 GEOMETRY Find the area of the rectangle. $8\sqrt{6} + 3\sqrt{2}$ ft^2

36. GEOMETRY Find the exact surface area of a sphere with radius of $4 + \sqrt{5}$ inches. $(84 + 32\sqrt{5})\pi$ in^2

Examples 5–6 Simplify. **37.** $56\sqrt{3} + 42\sqrt{6} - 36\sqrt{2} - 54$ **40.** $36\sqrt{2} + 36\sqrt{6} + 20\sqrt{3} + 60$

37. $(7\sqrt{2} - 3\sqrt{3})(4\sqrt{6} + 3\sqrt{12})$ **38.** $(8\sqrt{5} - 6\sqrt{3})(8\sqrt{5} + 6\sqrt{3})$ 212

39. $(12\sqrt{10} - 6\sqrt{5})(12\sqrt{10} + 6\sqrt{5})$ 1260 **40.** $(6\sqrt{3} + 5\sqrt{2})(2\sqrt{6} + 3\sqrt{8})$

42. $\dfrac{\sqrt{10} + \sqrt{6}}{2}$
43. $\dfrac{20 - 7\sqrt{3}}{11}$

41. $\dfrac{6}{\sqrt{3} - \sqrt{2}}$ $6\sqrt{3} + 6\sqrt{2}$ **42.** $\dfrac{\sqrt{2}}{\sqrt{5} - \sqrt{3}}$ **43.** $\dfrac{9 - 2\sqrt{3}}{\sqrt{3} + 6}$ **44.** $\dfrac{2\sqrt{2} + 2\sqrt{5}}{\sqrt{5} + \sqrt{2}}$ 2

22. $\dfrac{\sqrt{5a^5}}{\sqrt{b^{13}}}$ $\dfrac{a^2\sqrt{5ab}}{b^7}$

23. $\sqrt{\dfrac{7x}{10y^3}}$ $\dfrac{\sqrt{70xy}}{10y^2}$

24. $\dfrac{\sqrt[3]{6x^2}}{\sqrt[3]{5y}}$ $\dfrac{\sqrt[3]{150x^2y^2}}{5y}$

25. $\sqrt[4]{\dfrac{7x^3}{4b^2}}$ $\dfrac{\sqrt[4]{28b^2x^3}}{2|b|}$

25

$$\frac{\sqrt[4]{7x^3}}{\sqrt[4]{4b^2}} \cdot \frac{\sqrt[4]{4b^2}}{\sqrt[4]{4b^2}}$$

$2 \cdot 2 \cdot b \cdot b$

$$\frac{\sqrt[4]{28b^2x^3}}{\sqrt[4]{16b^4}}$$