

## What you'll learn about

- Indefinite Integrals
- Leibniz Notation and Antiderivatives
- Substitution in Indefinite Integrals
- Substitution in Definite Integrals

### ... and why

Antidifferentiation techniques were historically crucial for applying the results of calculus.

## 6.2 Antidifferentiation by Substitution

As mentioned before, our focus will be to learn new tricks to help us integrate.

First, we will accept a mathematical symbol that will tell us to perform the antiderivative.

### DEFINITION Indefinite Integral

The family of all antiderivatives of a function  $f(x)$  is the **indefinite integral of  $f$  with respect to  $x$**  and is denoted by  $\int f(x)dx$ .

If  $F$  is any function such that  $F'(x) = f(x)$ , then  $\int f(x)dx = F(x) + C$ , where  $C$  is an arbitrary constant, called the **constant of integration**.

As in Chapter 5, the symbol  $\int$  is an **integral sign**, the function  $f$  is the **integrand** of the integral, and  $x$  is the **variable of integration**.

### EXAMPLE 1 Evaluating an Indefinite Integral

Evaluate  $\int (x^2 - \sin x) dx$ .

#### SOLUTION

Evaluating this definite integral is just like solving the differential equation  $dy/dx = x^2 - \sin x$ . Our past experience with derivatives leads us to conclude that

$$\int (x^2 - \sin x) dx = \frac{x^3}{3} + \cos x + C$$

(as you can check by differentiating).

*Now try Exercise 3.*

In Exercises 1–6, find the indefinite integral.

1.  $\int (\cos x - 3x^2) dx$   
 $\sin x - x^3 + C$

2.  $\int x^{-2} dx$   
 $-x^{-1} + C$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

3.  $\int \left(t^2 - \frac{1}{t^2}\right) dt$   
 $t^3/3 + t^{-1} + C$

4.  $\int \frac{dt}{t^2 + 1}$   
 $\tan^{-1} t + C$

5.  $\int (3x^4 - 2x^{-3} + \sec^2 x) dx$   
 $(3/5)x^5 + x^{-2} + \tan x + C$

6.  $\int (2e^x + \sec x \tan x - \sqrt{x}) dx$   
 $2e^x + \sec x - (2/3)x^{3/2} + C$

### Properties of Indefinite Integrals

$$\int k f(x) dx = k \int f(x) dx \quad \text{for any constant } k$$

$$\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx$$

### Power Formulas

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \quad \text{when } n \neq -1$$

$$\int u^{-1} du = \int \frac{1}{u} du = \ln |u| + C$$

(see Example 2)

### Trigonometric Formulas

$$\int \cos u du = \sin u + C$$

$$\int \sin u du = -\cos u + C$$

$$\int \sec^2 u du = \tan u + C$$

$$\int \csc^2 u du = -\cot u + C$$

$$\int \sec u \tan u du = \sec u + C$$

$$\int \csc u \cot u du = -\csc u + C$$

### Exponential and Logarithmic Formulas

$$\int e^u du = e^u + C$$

$$\int a^u du = \frac{a^u}{\ln a} + C$$

$$\int \ln u du = u \ln u - u + C \quad (\text{See Example 2})$$

$$\int \log_a u du = \int \frac{\ln u}{\ln a} du = \frac{u \ln u - u}{\ln a} + C$$

Remember that cheat sheet I gave you?

Now, do all of those *in reverse*.

That's what is being shown on the left...

## EXAMPLE 2 Verifying Antiderivative Formulas

Verify the antiderivative formulas:

$$(a) \int u^{-1} du = \int \frac{1}{u} du = \ln |u| + C$$

$$(b) \int \ln u du = u \ln u - u + C$$

### SOLUTION

We can verify antiderivative formulas by differentiating.

$$(a) \text{ For } u > 0, \text{ we have } \frac{d}{du} (\ln |u| + C) = \frac{d}{du} (\ln u + C) = \frac{1}{u} + 0 = \frac{1}{u}.$$

$$\text{For } u < 0, \text{ we have } \frac{d}{du} (\ln |u| + C) = \frac{d}{du} (\ln(-u) + C) = \frac{1}{-u} (-1) + 0 = \frac{1}{u}.$$

Since  $\frac{d}{du} (\ln |u| + C) = \frac{1}{u}$  in either case,  $\ln |u| + C$  is the general antiderivative of the function  $\frac{1}{u}$  on its entire domain.

$$(b) \frac{d}{du} (u \ln u - u + C) = 1 \cdot \ln u + u \left( \frac{1}{u} \right) - 1 + 0 = \ln u + 1 - 1 = \ln u.$$

*product rule!*  
**Now try Exercise 11.**

In Exercises 7–12, use differentiation to verify the antiderivative formula.

7.  $\int \csc^2 u \, du = -\cot u + C$       8.  $\int \csc u \cot u \, du = -\csc u + C$   
 $(-\cot u + C)' = -(-\csc^2 u) = \csc^2 u$

9.  $\int e^{2x} \, dx = \frac{1}{2}e^{2x} + C$       10.  $\int 5^x \, dx = \frac{1}{\ln 5} 5^x + C$        $\frac{d}{dx} e^u = e^u \cdot u'$

11.  $\int \frac{1}{1+u^2} \, du = \tan^{-1} u + C$       12.  $\int \frac{1}{\sqrt{1-u^2}} \, du = \sin^{-1} u + C$

9  $\frac{d}{dx} \left( \frac{1}{2} e^{2x} + C \right)$   
 $= \frac{1}{2} e^{2x} \cdot 2 = e^{2x}$



### EXAMPLE 3 Paying Attention to the Differential

Let  $f(x) = x^3 + 1$  and let  $u = x^2$ . Find each of the following antiderivatives in terms of  $x$ :

(a)  $\int f(x) dx$       (b)  $\int f(u) du$       (c)  $\int f(u) dx$

#### SOLUTION

(a)  $\int f(x) dx = \int (x^3 + 1) dx = \frac{x^4}{4} + x + C$

(b)  $\int f(u) du = \int (u^3 + 1) du = \frac{u^4}{4} + u + C = \frac{(x^2)^4}{4} + x^2 + C = \frac{x^8}{4} + x^2 + C$

(c)  $\int f(u) dx = \int (u^3 + 1) dx = \int ((x^2)^3 + 1) dx = \int (x^6 + 1) dx = \frac{x^7}{7} + x + C$

① plug in "u"  
② plug in "x"  
③ Antiderivative  
③ plug in "x"  
③ Now try Exercise 15.  
antiderivative

In Exercises 13–16, verify that  $\int f(u) du \neq \int f(u) dx$

13.  $f(u) = \sqrt{u}$  and  $u = x^2 (x > 0)$

14.  $f(u) = u^2$  and  $u = x^5$

15.  $f(u) = e^u$  and  $u = 7x$

16.  $f(u) = \sin u$  and  $u = 4x$

Handwritten work for Exercise 13:

$$\int f(u) dx = \int \sqrt{u} dx$$

$$= \int \sqrt{x^2} dx$$

$$= \int x dx$$

$$= \frac{1}{2} x^2 + C$$

$$\int f(u) du = \int \sqrt{u} du = \int u^{1/2} du$$

$$= \frac{2}{3} u^{3/2} + C$$

$$= \frac{2}{3} (x^2)^{3/2} + C$$

$$= \frac{2}{3} x^3 + C$$

The two results are shown to be unequal ( $\neq$ ).

## Substitution in Indefinite Integrals

A change of variables can often turn an unfamiliar integral into one that we can evaluate. The important point to remember is that it is *not sufficient* to change an integral of the form  $\int f(x) dx$  into an integral of the form  $\int g(u) dx$ . The differential matters. A complete substitution changes the integral  $\int f(x) dx$  into an integral of the form  $\int g(u) du$ .



## EXAMPLE 4 Using Substitution

Evaluate  $\int \sin x e^{\cos x} dx$ .

### SOLUTION

Let  $u = \cos x$ . Then  $du/dx = -\sin x$ , from which we conclude that  $du = -\sin x dx$ . We rewrite the integral and proceed as follows:

$$\begin{aligned}\int \sin x e^{\cos x} dx &= -\int (-\sin x) e^{\cos x} dx \\ &= -\int e^{\cos x} \cdot (-\sin x) dx \\ &= -\int e^u du \\ &= -e^u + C \\ &= -e^{\cos x} + C\end{aligned}$$

Chain rule in reverse...

check ...

$$\frac{d}{dx} \left( -e^{\cos x} + C \right)$$

$u = \cos x$   
 $u' = -\sin x$

$-e^{\cos x} (+\sin x)$

In Exercises 17–24, use the indicated substitution to evaluate the integral. Confirm your answer by differentiation.

17.  $\int \sin 3x \, dx, \quad u = 3x \quad -\frac{1}{3} \cos 3x + C$   $\underline{u} = 3x$   
 $\underline{du} = 3 \, dx$

18.  $\int x \cos(2x^2) \, dx, \quad u = 2x^2$    
 $\sec u \tan u$

19.  $\int \sec 2x \tan 2x \, dx, \quad u = 2x \quad \frac{1}{2} \sec 2x + C$

20.  $\int 28(7x - 2)^3 \, dx, \quad u = 7x - 2$

21.  $\int \frac{dx}{x^2 + 9}, \quad u = \frac{x}{3}$   $(1/3) \tan^{-1}(x/3) + C$

22.  $\int \frac{9r^2 \, dr}{\sqrt{1 - r^3}}, \quad u = 1 - r^3$

23.  $\int \left(1 - \cos \frac{t}{2}\right)^2 \sin \frac{t}{2} \, dt, \quad u = 1 - \cos \frac{t}{2} \quad \frac{2}{3} \left(1 - \cos \frac{t}{2}\right)^3 + C$

24.  $\int 8(y^4 + 4y^2 + 1)^2 (y^3 + 2y) \, dy, \quad u = y^4 + 4y^2 + 1$

$\frac{1}{3} \int \sin 3x \, dx$   
 $= \frac{1}{3} \int \sin u \, du$   
 $= \frac{1}{3} (-\cos u) + C$   
 $= -\frac{1}{3} \cos 3x + C$

$$21. \int \frac{dx}{x^2 + 9}, \quad u = \frac{x}{3} \quad (1/3) \tan^{-1}(x/3) + C$$

$$\tan^{-1}(u) \quad u = \frac{x}{3} = \frac{1}{3}x$$

$$du = \frac{1}{3} dx$$

$$\frac{d}{dx} (\tan^{-1} x) = \frac{1}{1 + x^2}$$

$$23. \int \left(1 - \cos \frac{t}{2}\right)^2 \sin \frac{t}{2} dt, \quad u = 1 - \cos \frac{t}{2}$$

$$u = 1 - \cos \frac{t}{2}$$

$$du = \sin \frac{t}{2} \cdot \frac{1}{2} dt$$

$$2 \int \frac{1}{2} (1 - \cos \frac{t}{2}) \sin \frac{t}{2} dt$$

$$2 \int u' du = 2 \left( \frac{u^2}{2} \right)$$
$$= u^2$$
$$= \left(1 - \cos \frac{t}{2}\right)^2$$

### EXAMPLE 5 Using Substitution

Evaluate  $\int x^2 \sqrt{5 + 2x^3} dx$ .

$u = x^2$   
 $du = 2x \dots ?$

#### SOLUTION

This integral invites the substitution  $u = 5 + 2x^3$ ,  $du = 6x^2 dx$ .

$$\int x^2 \sqrt{5 + 2x^3} dx = \int (5 + 2x^3)^{1/2} \cdot x^2 dx$$

$$= \frac{1}{6} \int (5 + 2x^3)^{1/2} \cdot 6x^2 dx$$

$$= \frac{1}{6} \int u^{1/2} du$$

$$= \frac{1}{6} \left( \frac{2}{3} \right) u^{3/2} + C$$

$$= \frac{1}{9} (5 + 2x^3)^{3/2} + C$$

Set up the substitution with a factor of 6.

Substitute  $u$  for  $5 + 2x^3$  and  $du$  for  $6x^2 dx$ .

Re-substitute after antidifferentiating.

**Now try Exercise 27.**



### EXAMPLE 6 Using Substitution

Evaluate  $\int \cot 7x \, dx$ .

$\frac{u'}{u}$  look for this pattern!

$$\int \cot 7x \, dx = \int \frac{\cos 7x}{\sin 7x} \, dx$$

Trigonometric identity

$$= \frac{1}{7} \int \frac{7 \cos 7x \, dx}{\sin 7x}$$

Note that  $du = 7 \cos 7x \, dx$  when  $u = \sin 7x$

We multiply by  $\frac{1}{7} \cdot 7$ , or 1.

$$= \frac{1}{7} \int \frac{du}{u}$$

Substitute  $u$  for  $\sin 7x$  and  $du$  for  $7 \cos 7x \, dx$ .

$$= \frac{1}{7} \ln |u| + C$$

Notice the absolute value!

$$= \frac{1}{7} \ln |\sin 7x| + C$$

Re-substitute  $\sin 7x$  for  $u$  after antidifferentiating.

**Now try Exercise 29.**

In Exercises 25–46, use substitution to evaluate the integral.

25.  $\int \frac{dx}{(1-x)^2} = \frac{1}{1-x} + C$

25

$u = 1-x$   
 $du = -dx$   
 $-\int \frac{-dx}{(1-x)^2}$

27.  $\int \sqrt{\tan x} \sec^2 x dx = \frac{2}{3}(\tan x)^{3/2} + C$

28.  $\int \sec\left(\theta + \frac{\pi}{2}\right) \tan\left(\theta + \frac{\pi}{2}\right) d\theta$

[ ]

29.  $\int \tan(4x + 2) dx$

30.  $\int 3(\sin x)^{-2} dx$

[ ]

29.  $-(1/4) \ln|\cos(4x + 2)| + C$  or  $(1/4) \ln|\sec(4x + 2)| + C$

27  $u = \tan x$   
 $du = \sec^2 x dx$

Put back in terms of  $x$   
 $\int \sqrt{u} du = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (\tan x)^{3/2} + C = \frac{2}{3} \sqrt{\tan x}$

$= -\int \frac{du}{u^2}$   
 $= -\int u^{-2} du$   
 $= -(-u^{-1}) = \frac{1}{u} = \frac{1}{1-x}$

$$29. \int \tan(4x + 2) dx$$

$$u = \cos(4x + 2)$$

$$\left\{ \frac{\sin(4x + 2)}{\cos(4x + 2)} \right\} \left\{ \begin{array}{l} du = -\sin(4x + 2) \cdot 4 \\ -\frac{1}{4} du = \sin(4x + 2) \end{array} \right.$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$= \int \frac{-1}{4} \left( \frac{1}{u} \right) du$$

$$= -\frac{1}{4} (\ln|u|) + C$$

$$= -\frac{1}{4} (\ln|\cos(4x + 2)|) + C$$

$$31. \int \cos(3z + 4) dz \quad \left. \begin{array}{l} 3 \\ \text{NA} \end{array} \right\}$$

$$\frac{1}{3} \sin(3z + 4) + C$$

$$33. \int \frac{\ln^6 x}{x} dx \quad \left. \begin{array}{l} 1 \\ \text{NA} \end{array} \right\}$$

$$\frac{1}{7} (\ln x)^7 + C$$

$$35. \int s^{1/3} \cos(s^{4/3} - 8) ds \quad \left. \begin{array}{l} 3 \\ \text{NA} \end{array} \right\}$$

$$\frac{3}{4} \sin(s^{4/3} - 8) + C$$

$$37. \int \frac{\sin(2t + 1)}{\cos^2(2t + 1)} dt \quad \left. \begin{array}{l} 2 \\ \text{NA} \end{array} \right\}$$

$$(1/2) \sec(2t + 1) + C$$

$$39. \int \frac{dx}{x \ln x} \quad \left. \begin{array}{l} 1 \\ \text{NA} \end{array} \right\}$$

$$\ln(\ln x) + C$$

$$41. \int \frac{x dx}{x^2 + 1} \quad \left. \begin{array}{l} 42. \\ \text{NA} \end{array} \right\}$$

$$(1/2) \ln(x^2 + 1) + C$$

31

$$u = 3z + 4$$

$$du = 3 dz$$

$$\frac{1}{3} du = dz$$

$$\int \frac{1}{3} \cos(u) du$$

$$= -\frac{1}{3} \sin(3z + 4) + C$$



$$37. \int \frac{\sin(2t+1)}{\cos^2(2t+1)} dt$$

(1/2) sec(2t+1) + C

$$\frac{\sin \theta}{\cos \theta} \cdot \frac{1}{\cos \theta}$$

$$\int \tan(2t+1) \sec(2t+1) dt$$

$$u = 2t+1$$

$$du = 2 dt$$

$$\frac{1}{2} du = dt$$

$$\int \frac{1}{2} \tan u \sec u du$$

$$= \frac{1}{2} \sec(2t+1) + C$$



$$33. \int \frac{\ln^6 x}{x} dx = \frac{1}{7} (\ln x)^7 + C$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

$$\int \ln^6 x \cdot \frac{1}{x} dx \quad u = \ln x$$

$$du = \frac{1}{x} dx$$

$$= \int u^6 du = \frac{1}{7} u^7 + C = \frac{1}{7} (\ln x)^7 + C$$

$$33. \int \frac{\ln^6 x}{x} dx = \frac{1}{7} (\ln x)^7 + C$$

$$u = \frac{1}{x}$$
$$= x^{-1}$$

$$du = -x^{-2} dx$$

??

I need to

substitute...

... that should

be easier...



$$35. \int s^{1/3} \cos(s^{4/3} - 8) ds$$

$$\frac{3}{4} \sin(s^{4/3} - 8) + C$$

$$u = s^{4/3} - 8$$

$$du = \frac{4}{3} s^{1/3} ds$$

$$\frac{3}{4} du = s^{1/3} ds$$

$$\int \frac{3}{4} \cos(u) du$$

$$= \frac{3}{4} \sin(s^{4/3} - 8) + C$$

39.  $\int \frac{dx}{x \ln x}$

41.  $\int \frac{x dx}{x^2 + 1}$

(39)

$$u = \ln x \quad \left\{ \begin{array}{l} \frac{d}{dx} [\ln u] = \\ \frac{1}{u} \cdot u' \end{array} \right. = \frac{u'}{u}$$
$$du = \frac{1}{x} dx$$

$$\int \frac{du}{u} = \int \frac{1}{u} du$$
$$= \ln u + C$$
$$= \ln |\ln x| + C$$

$$u = x^2 + 1$$
$$du = 2x dx$$

(41)

$$\int \frac{2x dx}{x^2 + 1}$$

$$\int \frac{du}{u}$$
$$= \ln u$$
$$= \ln |x^2 + 1|$$

$$39. \int \frac{dx}{x \ln x}$$

$$41. \int \frac{x dx}{x^2 + 1}$$

$$u = \ln x$$

$$du = \frac{1}{x} dx$$

$$= \int \frac{du}{u} = \int \frac{1}{u} \cdot du$$

$$= \ln u$$

$$= \ln(\ln x) + C$$



$$41. \int \frac{x dx}{x^2 + 1}$$

$$= \int \frac{1}{2} \cdot \frac{1}{u} du$$

$$\begin{aligned} u &= x^2 + 1 \\ \frac{du}{2} &= \frac{2x dx}{2} \end{aligned}$$

$$\frac{1}{2} du = \underline{x dx}$$

$$= \frac{1}{2} \int \frac{1}{u} du$$

$$= \frac{1}{2} \ln|u| + C$$

$$= \frac{1}{2} \ln|x^2 + 1| + C$$

### EXAMPLE 7 Setting Up a Substitution with a Trigonometric Identity

Find the indefinite integrals. In each case you can use a trigonometric identity to set up a substitution.

(a)  $\int \frac{dx}{\cos^2 2x}$     (b)  $\int \cot^2 3x \, dx$     (c)  $\int \cos^3 x \, dx$

#### SOLUTION

$$\begin{aligned} \text{(a)} \int \frac{dx}{\cos^2 2x} &= \int \sec^2 2x \, dx \\ &= \frac{1}{2} \int \sec^2 2x \cdot 2 \, dx \\ &= \frac{1}{2} \int \sec^2 u \, du \\ &= \frac{1}{2} \tan u + C \\ &= \frac{1}{2} \tan 2x + C \end{aligned}$$

Let  $u = 2x$  and  $du = 2 \, dx$ .

Re-substitute after antidifferentiating.

$$\frac{d}{dx} [\tan u] = \sec^2 u \cdot u'$$

45.  $\int \sec x \, dx$  (Hint: Multiply the integrand by

$$\frac{\sec x + \tan x}{\sec x + \tan x} \quad \ln|\sec x + \tan x| + C$$

$$\int \left[ \frac{\sec x}{1} \left( \frac{\sec x + \tan x}{\sec x + \tan x} \right) \right] dx$$
$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

$$u = \sec x + \tan x$$
$$du = \sec x \tan x + \sec^2 x$$

$$= \int \frac{1}{u} du = \ln|u| + C$$

### EXAMPLE 7 Setting Up a Substitution with a Trigonometric Identity

Find the indefinite integrals. In each case you can use a trigonometric identity to set up a substitution.

(a)  $\int \frac{dx}{\cos^2 2x}$       (b)  $\int \cot^2 3x \, dx$       (c)  $\int \cos^3 x \, dx$

$$\begin{aligned}
 \text{(b)} \int \cot^2 3x \, dx &= \int (\underline{\csc^2 3x} - 1) \, dx \\
 &= \frac{1}{3} \int (\csc^2 3x - 1) \cdot 3 \, dx \\
 &= \frac{1}{3} \int (\csc^2 u - 1) \cdot du \quad \text{Let } u = 3x \text{ and } du = 3dx. \\
 &= \frac{1}{3} \int (-\cot u - u) + C \\
 &= \frac{1}{3} (-\cot 3x - 3x) + C \\
 &= -\frac{1}{3} \cot 3x - x + C \quad \text{Re-substitute after antidifferentiation.}
 \end{aligned}$$

Handwritten notes in red:

$$\frac{\sin^2 x + \cos^2 x = 1}{\sin^2 x} \quad \frac{\sin^2 x + \cos^2 x = 1}{\sin^2 x}$$

$$1 + \cot^2 x = \csc^2 x$$

$$\cot^2 x = \csc^2 x - 1$$

### EXAMPLE 7 Setting Up a Substitution with a Trigonometric Identity

Find the indefinite integrals. In each case you can use a trigonometric identity to set up a substitution.

(a)  $\int \frac{dx}{\cos^2 2x}$     (b)  $\int \cot^2 3x dx$     (c)  $\int \cos^3 x dx$

$$(c) \int \cos^3 x dx = \int (\cos^2 x) \cos x dx$$

$$= \int (1 - \sin^2 x) \cos x dx$$

$$= \int (1 - u^2) du$$

Let  $u = \sin x$  and  $du = \cos x dx$ .

$$= u - \frac{u^3}{3} + C$$

$$= \sin x - \frac{\sin^3 x}{3} + C$$

Re-substitute after antidifferentiating.

**Now try Exercise 47.**

In Exercises 47–52, use the given trigonometric identity to set up a  $u$ -substitution and then evaluate the indefinite integral.

47.  $\int \sin^3 2x \, dx$ ,  $\sin^2 2x = 1 - \cos^2 2x$   $\frac{\cos^3 x}{3} - \frac{\cos x}{2} + C$

48.  $\int \sec^4 x \, dx$ ,  $\sec^2 x = 1 + \tan^2 x$

49.  $\int 2 \sin^2 x \, dx$ ,  $\cos 2x = 2 \sin^2 x - 1$   $x + \frac{\sin 2x}{2} + C$   
 $\cos 2x + 1 = 2 \sin^2 x$

50.  $\int 4 \cos^2 x \, dx$ ,  $\cos 2x = 1 - 2 \cos^2 x$

$\left\{ \begin{array}{l} \cos 2x + 1 = \frac{1}{2} \sin 2x + x + C \\ \cos 2x - 1 = -\frac{1}{2} \sin 2x + x + C \end{array} \right.$



$$47. \int \sin^3 2x \, dx, \quad \sin^2 2x = 1 - \cos^2 2x$$

$$= \int (\sin^2 2x)(\sin 2x) \, dx$$

$$= \int (1 - \cos^2 2x)(\sin 2x) \, dx$$

$$u = \cos 2x$$

$$du = -\sin 2x \cdot 2 \, dx$$

$$\frac{-1}{2} du = \sin 2x \, dx$$

$$= \frac{-1}{2} \int (1 - u^2) \, du$$

$$= \frac{-1}{2} \left( u - \frac{1}{3} u^3 \right) + C$$

$$\frac{\cos^3 x}{3} - \frac{\cos x}{2} + C$$

$$= -\frac{1}{2} u - \frac{1}{6} u^3$$

$$= -\frac{1}{2} \cos 2x - \frac{1}{6} \cos^3 2x + C$$

$$= -\frac{1}{2}$$

$$\cos 2x = \sin^2 x - \cos^2 x$$

### EXAMPLE 8 Evaluating a Definite Integral by Substitution

Evaluate  $\int_0^{\pi/3} \tan x \sec^2 x \, dx$ .

#### SOLUTION

Let  $u = \tan x$  and  $du = \sec^2 x \, dx$ .

Note also that  $u(0) = \tan 0 = 0$  and  $u(\pi/3) = \tan(\pi/3) = \sqrt{3}$ .

So

$$\begin{aligned} \int_0^{\pi/3} \tan x \sec^2 x \, dx &= \int_0^{\sqrt{3}} u \, du && \text{Substitute } u\text{-interval for } x\text{-interval.} \\ &= \left. \frac{u^2}{2} \right|_0^{\sqrt{3}} \\ &= \frac{3}{2} - 0 = \frac{3}{2} \end{aligned}$$

*Now try Exercise 55.*

In Exercises 53–66, make a  $u$ -substitution and integrate from  $u(a)$  to  $u(b)$ .

53.  $\int_0^3 \sqrt{y+1} \, dy$  14/3

55.  $\int_{-\pi/4}^0 \tan x \sec^2 x \, dx$  -1/2

C 57.  $\int_0^1 \frac{10\sqrt{\theta}}{(1+\theta^{3/2})^2} \, d\theta$

59.  $\int_0^1 \sqrt{t^5+2t}(5t^4+2) \, dt$

61.  $\int_0^7 \frac{dx}{x+2}$

63.  $\int_1^2 \frac{dt}{t-3}$

65.  $\int_{-1}^3 \frac{x \, dx}{x^2+1}$

53  $u = y + 1$   
 $du = 1 \, dy$

SO...  
 $u(0) = 0 + 1 = 1$   
 $u(3) = 3 + 1 = 4$

$\int_1^4 \sqrt{u} \, du$

$= \left[ \frac{2}{3} u^{3/2} \right]_1^4$   
 $= \frac{2}{3} (4)^{3/2} - \frac{2}{3} (1)^{3/2}$   
 $= \frac{2}{3} (8) - \frac{2}{3}$   
 $= \frac{16}{3} - \frac{2}{3}$

$\text{D}$

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53.  $\int_0^3 \sqrt{y+1} dy$  14/3

55.  $\int_{-\pi/4}^0 \tan x \sec^2 x dx$  -1/2

57.  $\int_0^1 \frac{10\sqrt{\theta}}{(1+\theta^{3/2})^2} d\theta$  10/3

59.  $\int_0^1 \sqrt{t^5+2t}(5t^4+2) dt$   $\frac{3}{4}$   
 $2\sqrt{3}$

61.  $\int_0^7 \frac{dx}{x+2}$  1.504

63.  $\int_1^2 \frac{dt}{t-3}$  -0.693

65.  $\int_{-1}^3 \frac{x dx}{x^2+1}$  0.805

$\frac{3}{2}x = 10 \quad x = \frac{20}{3}$

57

63

$\int_0^1 \frac{10\theta^{1/2}}{(1+\theta^{3/2})^2} d\theta \quad du = dt$

$u = 1 + \theta^{3/2}$   
 $du = \frac{3}{2} \theta^{1/2}$

$\frac{20}{3} du = 10\theta^{1/2}$

$u = 1 + 0^{3/2} = 1$

$u = 1 + 1^{3/2} = 1 + 1 = 2$

$u = t - 3$

$u = 2 - 3 = -1$

$u = 1 - 3 = -2$

$\int_{-2}^{-1} \frac{1}{u} du$   
 $\int_{-2}^{-1} \frac{1}{|u|} du$  evaluate (smiley face)

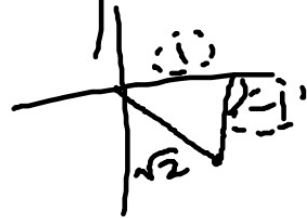
$\int_1^2 \frac{20}{3u^2} du$  evaluate (smiley face)

55.  $\int_{-\pi/4}^0 \tan x \sec^2 x dx$   $-1/2$

$u = \tan x$   
 $du = \sec^2 x dx$



SO...  
 $u(-\frac{\pi}{4}) = \tan(-\frac{\pi}{4})$   
 $= \frac{-1}{1} = -1$



$u(0) = \frac{0}{1} = 0$



$= \int_{-1}^0 u' du$   
 $= \left[ \frac{u^2}{2} \right]_{-1}^0 = \frac{0^2}{2} - \frac{(-1)^2}{2}$   
 $= -\frac{1}{2}$