let's add fractions!

$$\frac{(x+2)}{(x+2)}\frac{1}{x} + \frac{1}{x+2}\frac{(x)}{(x)}$$

$$\frac{x+2+x}{x(x+2)} = \frac{2x+2}{x^2+2x}$$

so....

$$\int \frac{1}{x}+\frac{1}{x+2} = \int \frac{2x+2}{x^2+2x}$$

$du=2x+2$

$u=x^2+2x$

?

$\ln x + \ln(x+2) =$

I want to break it down because it is easier.

6.5 Logistic Growth

In this section, we are learning how to break apart fractions, so that we could intergrate them easier.

Ex.

$$\frac{3}{10} = \frac{1}{10} + \frac{1}{5}$$

we will find this!

$2 \cdot 5$

Partial Fraction Decomposition with Distinct Linear Denominators

If $f(x) = \dfrac{P(x)}{Q(x)}$, where P and Q are polynomials with the degree of P less than the degree of Q, and if $Q(x)$ can be written as a product of distinct linear factors, then $f(x)$ can be written as a sum of rational functions with distinct linear denominators.

EXAMPLE 1 Finding a Partial Fraction Decomposition

Write the function $f(x) = \dfrac{x - 13}{2x^2 - 7x + 3}$ as a sum of rational functions with linear denominators.

SOLUTION

Since $f(x) = \dfrac{x - 13}{(2x - 1)(x - 3)}$, we will find numbers A and B so that

$$f(x) = \frac{A}{2x - 1} + \frac{B}{x - 3}. = \frac{x-13}{(2x-1)(x-3)}$$

$(2x-1)(x-3)$

$$A(x - 3) + B(2x - 1) = x - 13. \qquad (1)$$

Setting $x = 3$ in equation (1), we get

$$A(0) + B(5) = -10, \text{ so } B = -2.$$

Setting $x = \dfrac{1}{2}$ in equation (1), we get

$$A\left(-\frac{5}{2}\right) + B(0) = -\frac{25}{2}, \text{ so } A = 5.$$

Therefore $f(x) = \dfrac{x - 13}{(2x - 1)(x - 3)} = \dfrac{5}{2x - 1} - \dfrac{2}{x - 3}.$

Now try Exercise 3.

In Exercises 1–4, find the values of A and B that complete the partial fraction decomposition.

1. $\dfrac{x-12}{x^2-4x} = \dfrac{A}{x} + \dfrac{B}{x-4}$ $A = 3, B = -2$

2. $\dfrac{2x+16}{x^2+x-6} = \dfrac{A}{x+3} + \dfrac{B}{x-2}$ $A = -2, B = 4$

3. $\dfrac{16-x}{x^2+3x-10} = \dfrac{A}{x-2} + \dfrac{B}{x+5}$ $A = 2, B = -3$

4. $\dfrac{3}{x^2-9} = \dfrac{A}{x-3} + \dfrac{B}{x+3}$ $A = 1/2, B = -1/2$

1

LCD: $x(x-4)$

$$\frac{x-12}{x^2-4x} = \frac{A}{x} + \frac{B}{x-4}$$

$$x-12 = A(x-4) + Bx$$

$x=4$; $4-12 = A(4-4) + B(4)$

$-8 = 4B$

$B = -2$

$x=0$; $0-12 = A(0-4) + B(0)$

$-12 = -4A$

$A = 3$

$$= \frac{3}{x} - \frac{2}{x-4}$$

1. $\frac{x-12}{x^2-4x} = \frac{A}{x} + \frac{B}{x-4}$

LCD: x^2-4x

$\frac{x(x-4)}{1}$

$x=4$ $\qquad$ $x=0$

$$x-12 = A(x-4) + B(x)$$

$$4-12 = A(4-4) + B(4)$$

$$-8 = A(0) + 4B$$

$$\frac{-8}{4} = \frac{4B}{4} \qquad \boxed{B=-2}$$

$$0-12 = A(0-4) + B(0)$$

$$\frac{-12}{-4} = \frac{-4A}{-4} \qquad \boxed{A=3}$$

EXAMPLE 2 Antidifferentiating with Partial Fractions

Find $\displaystyle\int \frac{3x^4 + 1}{x^2 - 1}\,dx$. ⟵ only do this when it's bigger on top!

$$\frac{d}{dx}\left[\ln|u|\right] = \frac{u'}{u}$$

SOLUTION

First we note that the degree of the denominator is not less than the degree of the numerator. We use the division algorithm to find the quotient and remainder:

$$\begin{array}{r} 3x^2 + 3 \\ x^2 - 1 \,\overline{)\,3x^4 \qquad\quad + 1} \\ \underline{3x^4 - 3x^2 \qquad} \\ 3x^2 + 1 \\ \underline{3x^2 - 3} \\ 4 \end{array}$$

Thus

$$\int \frac{3x^4 + 1}{x^2 - 1}\,dx = \int \left(3x^2 + 3 + \frac{4}{x^2 - 1}\right) dx$$

continued

Thus

$$\int \frac{3x^4+1}{x^2-1}\,dx = \int\left(3x^2+3+\frac{4}{x^2-1}\right)dx$$

Partial Fractions!

$$= x^3+3x+\int \frac{4}{(x-1)(x+1)}\,dx$$

$$= x^3+3x+\int\left(\frac{A}{x-1}+\frac{B}{x+1}\right)dx.$$

We know that $A(x+1)+B(x-1)=4$.

Setting $x=1$,

$$A(2)+B(0)=4, \text{ so } A=2.$$

$$\int \frac{u'}{u} = \ln|u|$$

Setting $x=-1$,

$$A(0)+B(-2)=4, \text{ so } B=-2.$$

Thus

$$\int \frac{3x^4+1}{x^2-1}\,dx = x^3+3x+\int\left(\frac{2}{x-1}+\frac{-2}{x+1}\right)dx$$

$$= x^3+3x+2\ln|x-1|-2\ln|x+1|+C$$

$$= x^3+3x+2\ln\left|\frac{x-1}{x+1}\right|+C.$$

Now try Exercise 7.

In Exercises 5–14, evaluate the integral.

5. $\displaystyle\int \frac{x-12}{x^2-4x}\,dx$ $\quad \ln\dfrac{|x|^3}{(x-4)^2}+C$

6. $\displaystyle\int \frac{2x+16}{x^2+x-6}\,dx$ $\quad \ln\dfrac{(x-2)^4}{(x+3)^2}+C$

7. $\displaystyle\int \frac{2x^3}{x^2-4}\,dx$ $\quad x^2+\ln(x^2-4)^4+C$

8. $\displaystyle\int \frac{x^2-6}{x^2-9}\,dx$ $\quad x+\ln\sqrt{\left|\dfrac{x-3}{x+3}\right|}+C$

9. $\displaystyle\int \frac{2\,dx}{x^2+1}$ $\quad 2\tan^{-1}x+C$

10. $\displaystyle\int \frac{3\,dx}{x^2+9}$ $\quad \tan^{-1}\left(\dfrac{x}{3}\right)+C$

11. $\displaystyle\int \frac{7\,dx}{2x^2-5x-3}$ $\quad \ln\left|\dfrac{x-3}{2x+1}\right|+C$

12. $\displaystyle\int \frac{1-3x}{3x^2-5x+2}\,dx$ $\quad \ln\dfrac{|3x+2|}{(x-1)^2}+C$

13. $\displaystyle\int \frac{8x-7}{2x^2-x-3}\,dx$ $\quad \ln(|x+1|^3|2x-3|)+C$

14. $\displaystyle\int \frac{5x+14}{x^2+7x}\,dx$ $\quad \ln(x^2|x+7|^3)+C$

In Exercises 15–18, solve the differential equation

8. $\displaystyle\int \frac{x^2-6}{x^2-9}\,dx$ $\quad x + \ln\sqrt{\left|\frac{x-3}{x+3}\right|} + C$

$$\frac{x^2-6}{(x+3)(x-3)} = \frac{A}{x+3} + \frac{B}{x-3}$$

$$x^2-6 = A(x-3) + B(x+3)$$

$x=3$, $3 = 6B$, $B = \frac{1}{2}$

$x=-3$, $3 = -6A$, $A = -\frac{1}{2}$ SO...

$$\int \frac{x^2-6}{x^2-9} = \int \frac{-1}{2(x+3)} + \frac{1}{2(x-3)}$$

$$= \frac{-1}{2}\ln|x+3| + \frac{1}{2}\ln(x-3) + C$$

$$= \ln(x+3)^{-1/2} + \ln|x-3|^{1/2} + C$$

$$= \ln|x+3|^{-1/2}|x-3|^{1/2} + C$$

$$= \ln\frac{(x-3)^{1/2}}{(x+3)^{1/2}} = \ln\sqrt{\left|\frac{x-3}{x+3}\right|} + C$$

$|2x+1|$

13. $\displaystyle\int \frac{8x-7}{2x^2-x-3}\,dx$ $= \displaystyle\int \frac{8x-7}{(2x-3)(x+1)}$

$\ln(|x+1|^3|2x-3|^2)+C$

LCD: $(2x-3)(x+1)$

$$\frac{8x-7}{(2x-3)(x+1)} = \frac{A}{2x-3} + \frac{B}{x+1}$$

$2x-3=0$

$+3 \quad +3$

$\frac{2x}{2} = \frac{3}{2}$

$x = \frac{3}{2}$

$8x-7 = A(x+1) + B(2x-3)$

$x = -1$

$-15 = -5B$

$B = 3$

$x = \frac{3}{2}$

$5 = \frac{5}{2}A$

$A = 2$

so...

$= \displaystyle\int \frac{2}{2x-3} + \frac{3}{x+1}$

$2\ln|2x-3| + 3\ln|x+1|$

5

5. $\displaystyle\int \frac{x-12}{x^2-4x}\,dx$

(5) $\displaystyle\int \frac{x-12}{x^2-4x} = \int \frac{x-12}{x(x-4)}$

LCD: $x(x-4)$

$$\frac{x-12}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4}$$

$x-12 = A(x-4) + Bx$

$x=4$; $-8 = 4B$ $B = -2$

$x=0$ $-12 = -4A$ $A = 3$

$$= \int \left(\frac{3}{x} - \frac{2}{x-4}\right) dx$$

$$= 3\ln|x| - 2\ln|x-4|$$

$$\ln \frac{|x|^3}{(x-4)^2} + C$$

In Exercises 5–14, evaluate the integral.

5. $\int \frac{x-12}{x^2-4x}\,dx$ $\ln\frac{|x|^3}{(x-4)^2}+C$

6. $\int \frac{2x+16}{x^2+x-6}\,dx$ $\ln\frac{(x-2)^4}{(x+3)^2}+C$

7. $\int \frac{2x^3}{x^2-4}\,dx$ ← use long division 1st!

$x^2+\ln(x^2-4)^4+C$

8. $\int \frac{x^2-6}{x^2-9}\,dx$ $x+\ln\sqrt{\left|\frac{x-3}{x+3}\right|}+C$

9. $\int \frac{2\,dx}{x^2+1}$ $2\tan^{-1}x+C$

← look up inverse trig!

10. $\int \frac{3\,dx}{x^2+9}$ $\tan^{-1}\left(\frac{x}{3}\right)+C$

11. $\int \frac{7\,dx}{2x^2-5x-3}$ $\ln\left|\frac{x-3}{2x+1}\right|+C$

13. $\int \frac{8x-7}{2x^2-x-3}\,dx$

$\ln(|x+1|^3|2x-3|)+C$

Decompose!

$$= \int 2x + \int \frac{8x}{x^2-4}$$

7 $x^2-4\overline{)2x^3+0x^2+0x+0}$

$-\cancel{2x^3} \quad -8x$

$8x$

11 $7\int \frac{1}{(2x-1)(x+3)}\,dx$

$$\frac{1}{(2x-1)(x+3)} = \frac{A}{(2x-1)} + \frac{B}{(x+3)}$$

$$1 = (x+3)A + (2x-1)B$$

if $x=-3$ ... if $x=\frac{1}{2}$ ...

$1=-7B$ | $1=\frac{7}{2}A$

$B=-1/7$ | $A=\frac{2}{7}$

$$7\int \frac{2}{7(2x-1)} - \frac{1}{7(x+3)}$$

$$\int \frac{2}{2x-1}\,dx - \int \frac{1}{x+3}\,dx$$

$u' \rightarrow 2$, $u \rightarrow 2x-1$; $1 \leftarrow u'$, $x+3 \leftarrow u$

$\int \frac{u'}{u} = \ln|u|$

$u=2x-1$

$u'=du=2\,dx$

$$= \ln|2x-1| - \ln|x+3|$$

$\ln|M| - \ln|N| = \ln\left|\frac{M}{N}\right|$

$(x-1)^2$

14. $\displaystyle\int \frac{5x+14}{x^2+7x}\,dx$

$\ln(x^2|x+7|^3)+C$

14 $\displaystyle\int \frac{5x+14}{x(x+7)} = \int \frac{A}{x} + \frac{B}{x+7} = \int \frac{2}{x} + \frac{3}{x+7}$

$5x+14 = A(x+7) + Bx$ $\quad = 2\ln x + 3\ln|x+7|$

$x=-7$

$-21 = -7B$

$B=3$

$x=0$

$14=7A$ $\quad = \ln x^2 + \ln|x+7|^3$

$A=2$ $\quad = \ln x^2|x+7|^3 + C$

so...

EXAMPLE 3 Finding Three Partial Fractions

This example will be our most laborious problem.

Find the general solution to $\dfrac{dy}{dx} = \dfrac{6x^2 - 8x - 4}{(x^2 - 4)(x - 1)}$.

SOLUTION

$$y = \int \frac{6x^2 - 8x - 4}{(x - 2)(x + 2)(x - 1)}\,dx = \int \left(\frac{A}{x - 2} + \frac{B}{x + 2} + \frac{C}{x - 1} \right) dx.$$

We know that $A(x + 2)(x - 1) + B(x - 2)(x - 1) + C(x - 2)(x + 2) = 6x^2 - 8x - 4$.

Setting $x = 2$:

$$A(4)(1) + B(0) + C(0) = 4, \text{ so } A = 1.$$

Setting $x = -2$:

$$A(0) + B(-4)(-3) + C(0) = 36, \text{ so } B = 3.$$

Setting $x = 1$,

$$A(0) + B(0) + C(-1)(3) = -6, \text{ so } C = 2.$$

Thus

$$\int \frac{6x^2 - 8x - 4}{(x - 2)(x + 2)(x - 1)}\,dx = \int \left(\frac{1}{x - 2} + \frac{3}{x + 2} + \frac{2}{x - 1} \right) dx$$

$$= \ln|x - 2| + 3\ln|x + 2| + 2\ln|x - 1| + C$$

$$= \ln\left(|x - 2||x + 2|^3\,|x - 1|^2\right) + C.$$

Now try Exercise 17.

In Exercises 15–18, solve the differential equation.

15. $\dfrac{dy}{dx} = \dfrac{2x-6}{x^2-2x}$

16. $\dfrac{du}{dx} = \dfrac{2}{x^2-1}$

17. $F'(x) = \dfrac{2}{x^3-x}$

$x(x^2-1)$

18. $G'(t) = \dfrac{2t^3}{t^3-t}$

$t(t^2-1)$

Answers:

15. $y = \ln\left|\dfrac{x^3}{x-2}\right| + C$ **16.** $u = \ln\left|\dfrac{x-1}{x+1}\right| + C$ **17.** $F(x) = \ln\dfrac{|x^2-1|}{x^2} + C$ **18.** $G(x) = 2x + \ln\left|\dfrac{x-1}{x+1}\right| + C$

15 $\dfrac{dy}{dx} = \dfrac{2x-6}{x(x-2)} = \dfrac{A}{x} + \dfrac{B}{x-2}$

$2x-6 = A(x-2) + B(x)$

$x=2:$
$-2 = 2B$
$B=-1$

$x=0$
$-6 = -2A$
$A = 3$
so...

$\ln A - \ln B = \ln \dfrac{A}{B}$

$y = \int \dfrac{3}{x} - \dfrac{1}{x-2}\,dx$

$= 3\ln|x| - \ln|x-2|$

$= \ln|x|^3 - \ln|x-2|$

$= \ln\left|\dfrac{x^3}{x-2}\right| + C$

18. $G'(t) = \dfrac{2t^3}{t^3 - t}$

$t(t^2-1)$

$\left| + C\right.$ **17.** $F(x) = \ln\dfrac{|x^2-1|}{x^2} + C$ **18.** $G(x) = 2x + \ln\left|\dfrac{x-1}{x+1}\right| + C$

Partial fractions!

$$2 + \frac{2t}{t^3-t}$$

18

$$t^3 - t \,\big)\, 2t^3 + 0t^2 + 0t + 0$$
$$-\left(2t^3 \quad -2t\right)$$
$$2t$$

$$\frac{2t}{t(t+1)(t-1)} = \frac{A}{t} + \frac{B}{t+1} + \frac{C}{t-1}$$

$$2t = A(t+1)(t-1) + B(t)(t-1) + C(t)(t+1)$$

$t = 0;$ $0 = -A$, $A = 0$

$t = -1;$ $-2 = 2B$, $B = -1$

$t = 1;$ $2 = 2C$, $C = 1$

so

$$\int \frac{2t^3}{t^3-1} = \int 2 - \frac{1}{t+1} + \frac{1}{t-1}\,dx$$

$$= 2x + \ln|t+1|^{-1} + \ln|t-1| + C$$

$$= 2x + \ln\left|\frac{t-1}{t+1}\right| + C$$

In Exercises 19–22, find the integral *without* using the technique of partial fractions.

19. $\displaystyle\int \frac{2x}{x^2 - 4}\,dx$ $\quad \ln|x^2 - 4| + C$

20. $\displaystyle\int \frac{4x - 3}{2x^2 - 3x + 1}\,dx$ $\quad \ln|2x^2 - 3x + 1| + C$

21. $\displaystyle\int \frac{x^2 + x - 1}{x^2 - x}\,dx$

$x + \ln|x^2 - x| + C$

22. $\displaystyle\int \frac{2x^3}{x^2 - 1}\,dx$

$x^2 + \ln|x^2 - 1| + C$