7-8 Using Exponential and Logarithmic Functions

KeyConcept Exponential Growth and Decay

Exponential Growth	Exponential Decay
Exponential growth can be modeled by the function $f(x) = ae^{kt}$, where a is the initial value, t is time in years, and k is a constant representing the rate of continuous growth.	Exponential decay can be modeled by the function $f(x) = ae^{-kt}$, where a is the initial value, t is time in years, and k is a constant representing the rate of continuous decay.

Real-World Example 1 Exponential Decay

SCIENCE The half-life of a radioactive substance is the time it takes for half of the atoms of the substance to disintegrate. The half-life of Carbon-14 is 5730 years. Determine the value of k and the equation of decay for Carbon-14.

If a is the initial amount of the substance, then the amount y that remains after 5730 years can be represented by $\frac{1}{2}a$ or $0.5a$.

$y = ae^{-kt}$	Exponential Decay Formula
$0.5a = ae^{-k(5730)}$	$y = 0.5a$ and $t = 5730$
$0.5 = e^{-5730k}$	Divide each side by a.
$\ln 0.5 = \ln e^{-5730k}$	Property of Equality for Logarithmic Functions
$\ln 0.5 = -5730k$	$\ln e^x = x$
$\frac{\ln 0.5}{-5730} = k$	Divide each side by -5730.
$0.00012 \approx k$	Use a calculator.

Thus, the equation for the decay of Carbon-14 is $y = ae^{-0.00012t}$.

① As the exponent gets bigger...

$$10 \cdot 2^{1} = 20$$

start w/ this...

$$10 \cdot 2^{2} = 40$$

the value gets bigger...

(2)

As the exponent decreases...

$$10 \cdot 2^{-1} = 10 \cdot \frac{1}{2} = 5$$

start with this...

$$10 \cdot 2^{-2} = \frac{10}{4} = 2.5$$

the value decreases...

a. How long ago did the animal live?

Understand The formula for the decay of Carbon-14 is $y = ae^{-0.00012t}$. You
out how long ago the animal lived.

Plan Let a be the initial amount of Carbon-14 in the animal's body.
y that remains after t years is 2% of a or $0.02a$.

Solve

$y = ae^{-0.00012t}$	Formula for the decay of Carbon-14
$0.02a = ae^{-0.00012t}$	$y = 0.02a$
$0.02 = e^{-0.00012t}$	Divide each side by a.
$\ln 0.02 = \ln e^{-0.00012t}$	Property of Equality for Logarithmic Functi
$\ln 0.02 = -0.00012t$	$\ln e^x = x$
$\dfrac{\ln 0.02}{-0.00012} = t$	Divide each side by -0.00012.
$32{,}600 \approx t$	Use a calculator.

The animal lived about 32,600 years ago.

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Real-World Example 2 Carbon Dating

SCIENCE **A paleontologist examining the bones of a prehistoric animal estimates that they contain 2% as much Carbon-14 as they would have contained when the animal was alive.**

b. If prior research points to the animal being around 20,000 years old, how much Carbon-14 should be in the animal?

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Examples 1–2

1. PALEONTOLOGY The half-life of Potassium-40 is about 1.25 billion years.

a. Determine the value of k and the equation of decay for Potassium-40. 5.545×10^{-10} $= k$

b. A specimen currently contains 36 milligrams of Potassium-40. How long will it take the specimen to decay to only 15 milligrams of Potassium-40? **1,578,843,530 yr**

c. How many milligrams of Potassium-40 will be left after 300 million years?

d. How long will it take Potassium-40 to decay to one eighth of its original amount?

$$.5a = ae^{-k(1.25)}$$

$$\frac{.5a}{a} = \frac{ae^{-1.25k}}{a}$$

$$.5 = e^{-1.25k}$$

$$\ln .5 = \ln e^{-1.25k}$$

$$\frac{\ln(.5)}{-1.25} = \frac{-1.25k}{-1.25}$$

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Real-World Example 3 Continuous Exponential Growth

POPULATION **In 2007, the population of the state of Georgia was 9.36 million people. In 2000, it was 8.18 million.**

a. Determine the value of k, Georgia's relative rate of growth.

Examples 1–2 **1. PALEONTOLOGY** The half-life of Potassium-40 is about 1.25 billion years.

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c. How many milligrams of Potassium-40 will be left after 300 million years? **about 30.48 mg**

d. How long will it take Potassium-40 to decay to one eighth of its original amount? **3,750,120,003 yr**

(b)

$$y = a \cdot e^{-(5.545 \times 10^{-10})t}$$

$$15 = 36e^{-(5.545 \times 10^{-10})t}$$

$$\ln \frac{15}{36} = \ln e^{-(5.545 \times 10^{-10})t}$$

$$\frac{\ln\left(\frac{15}{36}\right)}{-5.545 \times 10^{-10}} = \frac{\left(-5.545 \times 10^{-10}\right)t}{-5.545 \times 10^{-10}}$$

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Real-World Example 3 Continuous Exponential Growth

POPULATION **In 2007, the population of the state of Georgia was 9.36 million people. In 2000, it was 8.18 million.**

b. When will Georgia's population reach 12 million people?

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Real-World Example 3 Continuous Exponential Growth

POPULATION In 2007, the population of the state of Georgia was 9.36 million people. In 2000, it was 8.18 million.

c. Michigan's population in 2000 was 9.9 million and can be modeled by $y = 9.9e^{0.0028t}$. Determine when Georgia's population will surpass Michigan's.

$8.18e^{0.01925t} > 9.9e^{0.0028t}$ — Formula for exponential growth

$\ln 8.18e^{0.01925t} > \ln 9.9e^{0.0028t}$ — Property of Inequality for Logarithms

$\ln 8.18 + \ln e^{0.01925t} > \ln 9.9 + \ln e^{0.0028t}$ — Product Property of Logarithms

$\ln 8.18 + 0.01925t > \ln 9.9 + 0.0028t$ — $\ln e^x = x$

$0.01645t > \ln 9.9 - \ln 8.18$ — Subtract $(0.0028t + \ln 8.18)$ from each side.

$t > \frac{\ln 9.9 - \ln 8.18}{0.01645}$ — Divide each side by 0.01645.

$t > 11.6$ — Use a calculator.

Georgia's population will surpass Michigan's during 2012.

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a) $8 = 2e^{20k}$

Example 3

2. **SCIENCE** A certain food is dropped on the floor and is growing bacteria exponentially according to the model $y = 2e^{kt}$, where t is the time in seconds. **b. about 2.828 cells**

   a. If there are 2 cells initially and 8 cells after 20 seconds, find the value of k for the bacteria. $k \approx 0.0693$

   b. The "5-second rule" says that if a person who drops food on the floor eats it within 5 seconds, there will be no harm. How much bacteria is on the food after 5 seconds?

   c. Would you eat food that had been on the floor for 5 seconds? Why or why not? Do you think that the information you obtained in this exercise is reasonable? Explain.

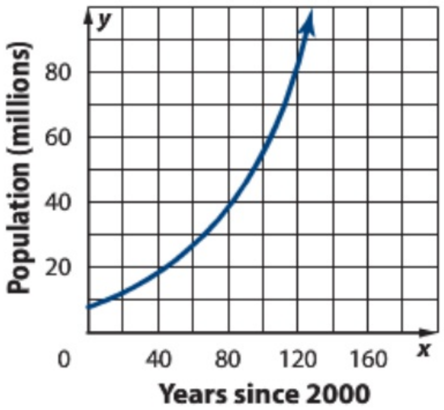
2c.

a) (cont.)

$4 = e^{20k}$

$\frac{\ln 4}{20} = \frac{20k}{20}$

2 Logistic Growth Refer to the equation representing Georgia's population in Example 3. According to the graph at the right, Georgia's population will be about one billion by the year 2130. Does this seem logical?



Populations cannot grow infinitely large. There are limitations, such as food supplies, war, living space, diseases, available resources, and so on.

Exponential growth is unrestricted, meaning it will increase without bound. A **logistic growth model**, however, represents growth that has a limiting factor. Logistic models are the most accurate models for representing population growth.

KeyConcept Logistic Growth Function

Let a, b, and c be positive constants where $b < 1$. The logistic growth function is represented by $f(t) = \frac{c}{1 + ae^{-bt}}$, where t represents time.

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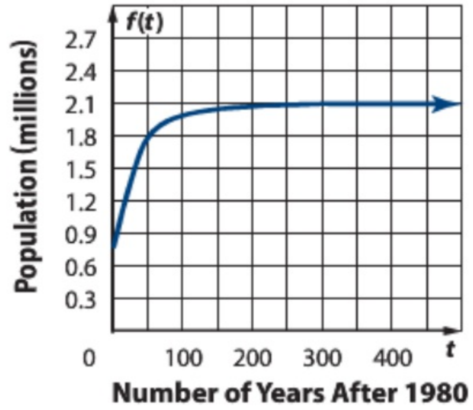
Real-World Example 4 Logistic Growth

The population of Phoenix, Arizona, in millions can be modeled by the logistic function $f(t) = \frac{2.0666}{1 + 1.66e^{-0.048t}}$, where t is the number of years after 1980.

a. Graph the function for $0 \leq t \leq 500$.



b. What is the horizontal asymptote?

c. Will the population of Phoenix increase indefinitely? If not, what will be their maximum population?

d. According to the function, when will the population of Phoenix reach 1.8 million people?

KeyConcept Logistic Growth Function

Let a, b, and c be positive constants where $b < 1$. The logistic growth function is represented by $f(t) = \frac{c}{1 + ae^{-bt}}$, where t represents time.

Example 4

3. ZOOLOGY Suppose the red fox population in a restricted habitat follows the function $P(t) = \frac{16{,}500}{1 + 18e^{-0.085t}}$, where t represents the time in years.

a. Graph the function for $0 \leq t \leq 200$. **See margin.**

b. What is the horizontal asymptote?

c. What is the maximum population?

d. When does the population reach 16,450?

Practice and Problem Solving

Extra Practice is on page R7.

Examples 1–2

4. CCSS **PERSEVERANCE** The half-life of Rubidium-87 is about 48.8 billion years.

a. Determine the value of k and the equation of decay for Rubidium-87. **$k \approx 1.42 \times 10^{-11}$**

b. A specimen currently contains 50 milligrams of Rubidium-87. How long will it take the specimen to decay to only 18 milligrams of Rubidium-87? **71,947,270,950 yr**

c. How many milligrams of Rubidium-87 will be left after 800 million years? **about 49.4 mg**

d. How long will it take Rubidium-87 to decay to one-sixteenth its original amount? **195.3 billion yr**

Example 3

5 **BIOLOGY** A certain bacteria is growing exponentially according to the model $y = 80e^{kt}$, where t is the time in minutes.

a. If there are 80 cells initially and 675 cells after 30 minutes, find the value of k for the bacteria. **$k \approx 0.071$**

b. When will the bacteria reach a population of 6000 cells? **about 60.8 min**

c. If a second type of bacteria is growing exponentially according to the model $y = 35e^{0.0978t}$, determine how long it will be before the number of cells of this bacteria exceed the number of cells in the other bacteria. **about 30.85 min**

Example 4

6. **FORESTRY** The population of trees in a certain forest follows the function $f(t) = \frac{18000}{1 + 16e^{-0.084t}}$, where t is the time in years.

a. Graph the function for $0 \leq t \leq 100$. **See margin.**

b. When does the population reach 17500 trees? **about 75.33 yr**

6a.