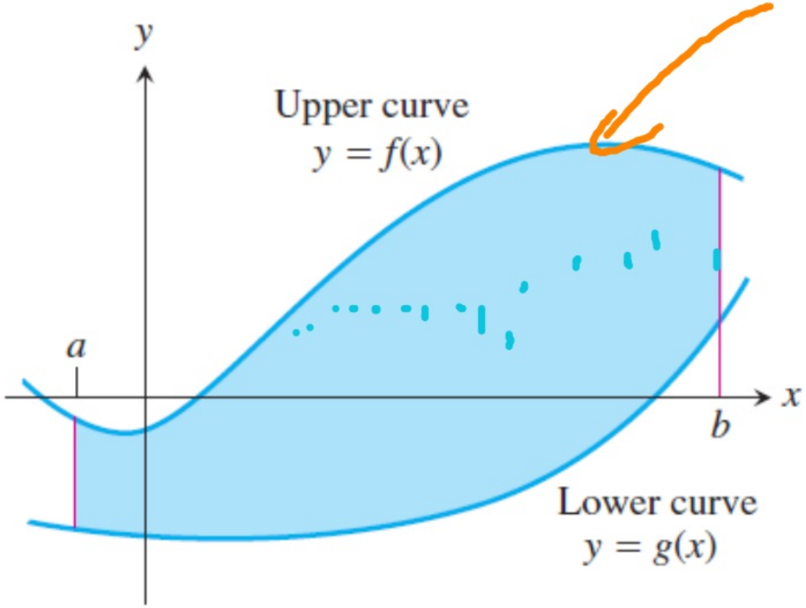
## What you'll learn about

- Area Between Curves
- Area Enclosed by Intersecting Curves
- Boundaries with Changing Functions
- Integrating with Respect to $y$

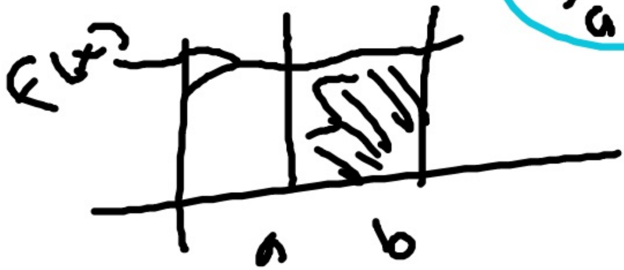
# 7.2 Areas in the Plane

We know how to find areas between a graph and the x axis, but what if we wanted to find the area between curves?

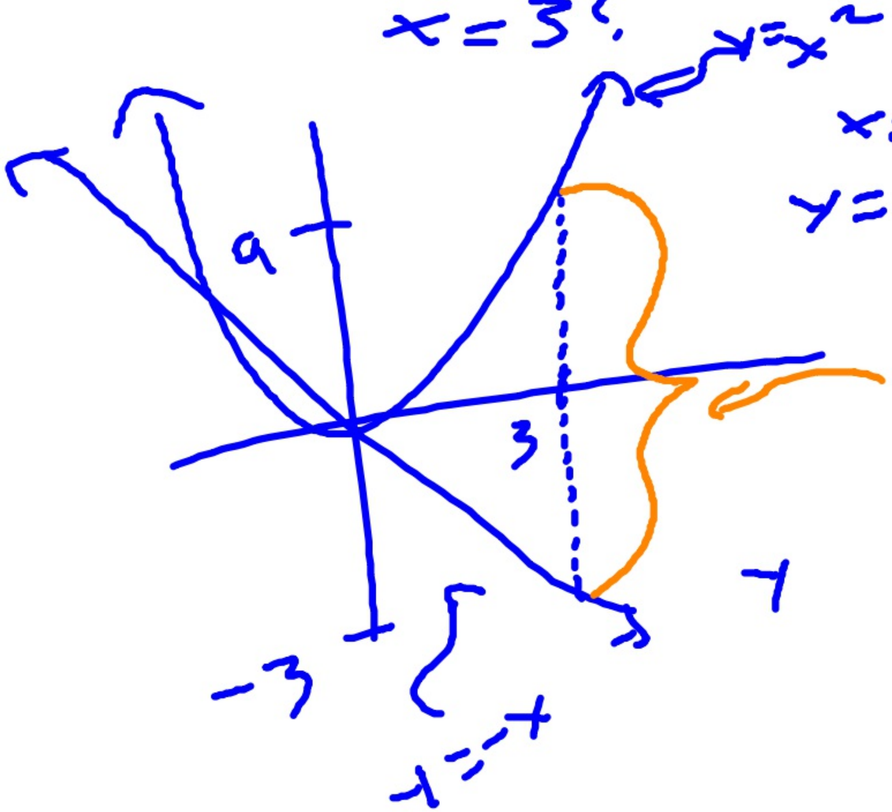



Before ....

$\int_a^b f(x)$



Q: What is the distance between $y = x^2$ and $y = -x$ when $x = 3$?




$x = 3 \ldots$

$y = 3^2 = 9$

$3 \quad (-9)$

$9 - (-3)$

$= 12$

Since we are finding a distance between two points along a number line (y-axis), we can subtract the two values to express the total distance.

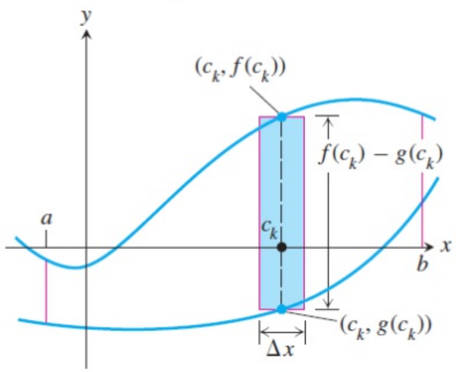



**Figure 7.5** The area of a typical rectangle is $[f(c_k) - g(c_k)]\,\Delta x$.

$$[f(c_k) - g(c_k)]\,\Delta x$$

(typical area of one rectangle)

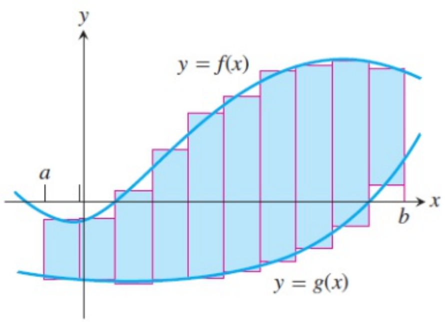



**Figure 7.4** We approximate the region with rectangles perpendicular to the $x$-axis.

$$\sum [f(c_k) - g(c_k)]\,\Delta x.$$

(typical sum of the area of rectangles, or region)

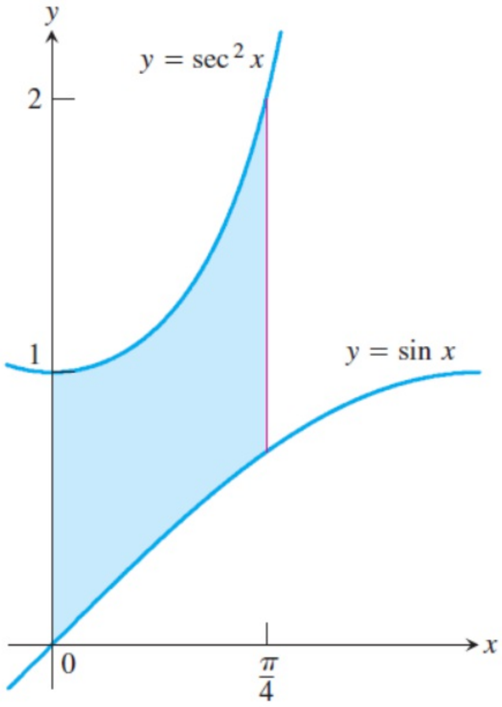
The limit of these sums as $\Delta x \to 0$ is

$$\int_a^b [f(x) - g(x)]\,dx.$$

## DEFINITION Area Between Curves

If $f$ and $g$ are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area between the curves $y = f(x)$ and $y = g(x)$ from $a$ to $b$** is the integral of $[f - g]$ from $a$ to $b$,

$$A = \int_a^b [f(x) - g(x)]\, dx.$$




## EXAMPLE 1 Applying the Definition

Find the area of the region between $y = \sec^2 x$ and $y = \sin x$ from $x = 0$ to $x = \pi/4$.

**SOLUTION**

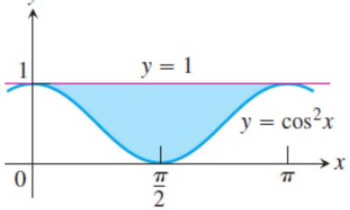
We graph the curves (Figure 7.6) to find their relative positions in the plane, and see that $y = \sec^2 x$ lies *above* $y = \sin x$ on $[0, \pi/4]$. The area is therefore

$$\begin{aligned} A &= \int_0^{\pi/4} [\sec^2 x - \sin x]\, dx \\ &= \Big[\tan x + \cos x\Big]_0^{\pi/4} \\ &= \frac{\sqrt{2}}{2} \text{ units squared.} \end{aligned}$$

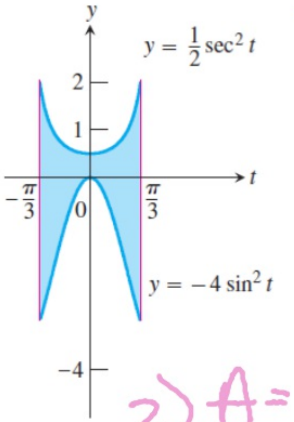
***Now try Exercise 1.***

In Exercises 1–6, find the area of the shaded region analytically.

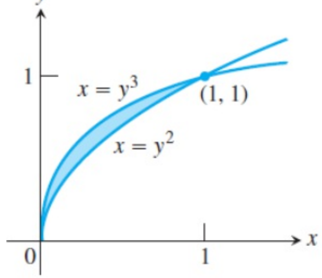
**1.** $\pi/2$



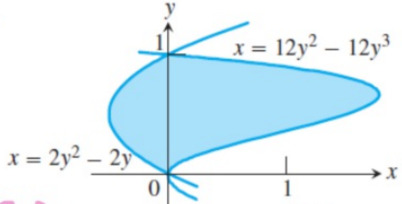

**2.** $4\pi/3$




**3.** 1/12




**4.** 4/3




2) $A = \int_{-\pi/3}^{\pi/3} \left[\left(\frac{1}{2}\sec^2 t\right) - \left(-4\sin^2 t\right)\right] dt$

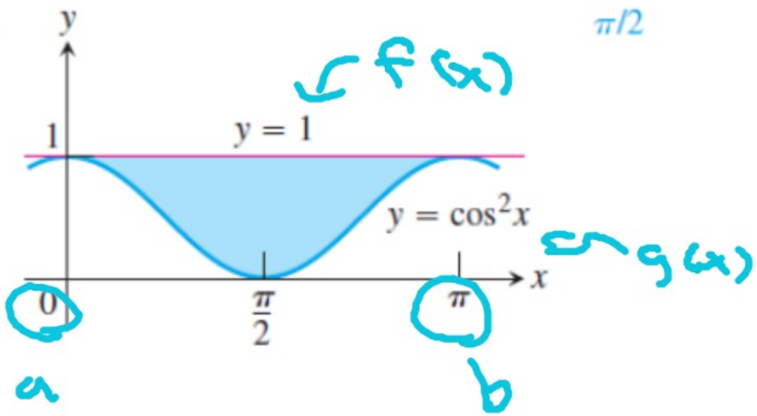
$= 4.18$

4

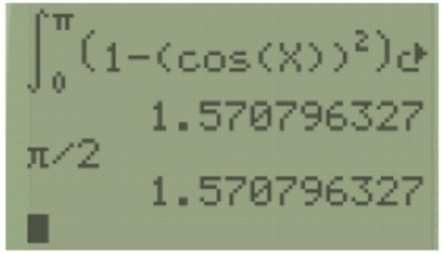
$A = \int_0^1 \left[\left(12y^2 - 12y^3\right) - \left(2y^2 - 2y\right)\right] dy$

$= 1.\overline{33}$

**1.**

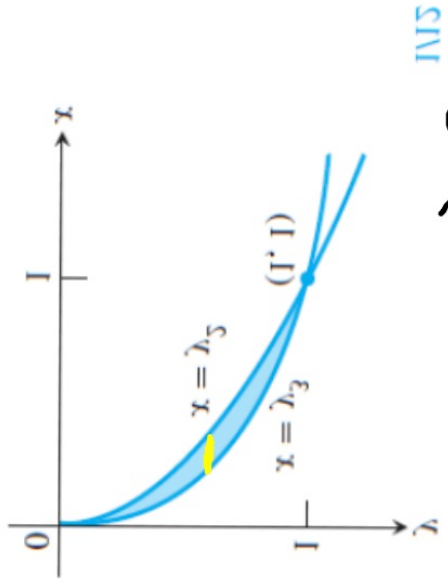



π/2




$$A = \int_0^{\pi} \left[1 - \cos^2 x\right] dx$$

$\frac{1}{12} = .08\overline{33}$



$$\int_0^1 (y^2 - y^3)\,dy$$

$\int_0^1 (y^2 - x^3)\,dX$

.0833333333

OR....

$\sqrt[3]{x} = \sqrt[3]{y^3}$

$\cancel{y} = \sqrt[3]{x}$

???

$$\int_0^1 y^3 - y^2\,dx \qquad \sqrt{y^2} = \sqrt{x}$$

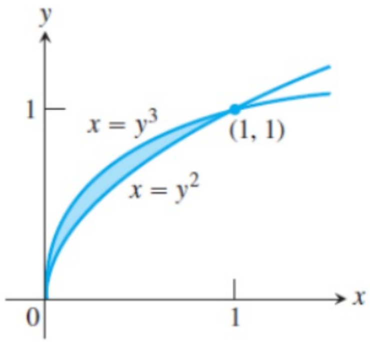
$y = \sqrt{x}$

$\int_0^1 (\sqrt[3]{X} - \sqrt{X})\,dX$

.0833340645

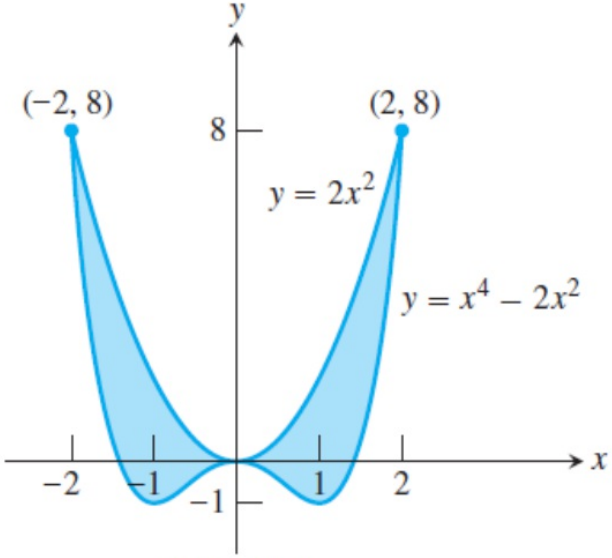
$$\int_0^1 (\sqrt[3]{x} - \sqrt{x})\,dx$$

**3.**




1/12

**5.**



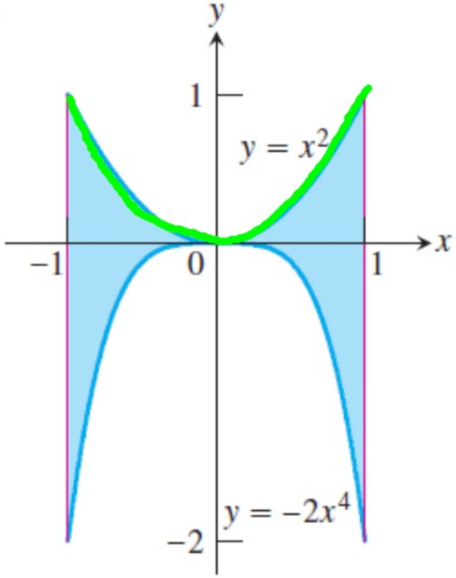

NOT TO SCALE

128/15

I don't have to break down the integration because one graph is *always* above the other graph.

$$\int_{-2}^{2}\left[2x^2-(x^4-2x^2)\right]dx$$

**6.**




22/15

$$\int_{-1}^{1}\left[x^2-(-2x^4)\right]dx$$

## EXAMPLE 2 Area of an Enclosed Region

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

**SOLUTION**

We graph the curves to view the region (Figure 7.8).

The limits of integration are found by solving the equation

$$2 - x^2 = -x$$

either algebraically or by calculator. The solutions are $x = -1$ and $x = 2$.

*continued*

Since the parabola lies above the line on $[-1, 2]$, the area integrand is $2 - x^2 - (-x)$.

$$A = \int_{-1}^{2} [2 - x^2 - (-x)]\, dx$$

$$= \left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{2}$$

$$= \frac{9}{2} \text{ units squared}$$

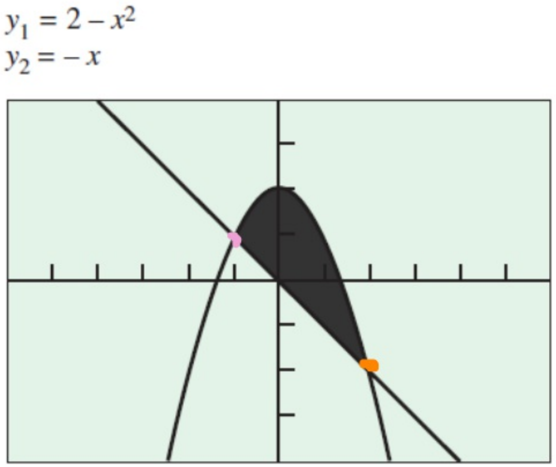



**Figure 7.8** The region in Example 2.

### EXAMPLE 3 Using a Calculator

Find the area of the region enclosed by the graphs of $y = 2\cos x$ and $y = x^2 - 1$.

**SOLUTION**

The region is shown in Figure 7.9.

Using a calculator, we solve the equation

$$2\cos x = x^2 - 1$$

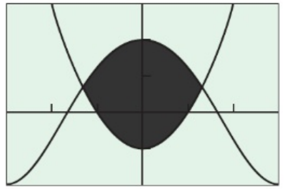
to find the $x$-coordinates of the points where the curves intersect. These are the limits of integration. The solutions are $x = \pm 1.265423706$. We store the negative value as $A$ and the positive value as $B$. The area is

$$\text{NINT}\,(2\cos x - (x^2 - 1), x, A, B) \approx 4.994907788.$$

This is the final calculation, so we are now free to round. The area is about 4.99.

*Now try Exercise 7.*

$y_1 = 2\cos x$
$y_2 = x^2 - 1$

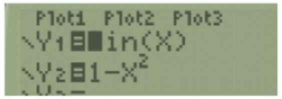


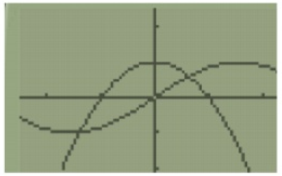
$[-3, 3]$ by $[-2, 3]$

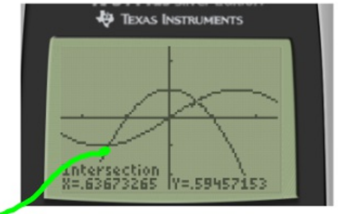
**Figure 7.9** The region in Example 3.

In Exercises 7 and 8, use a calculator to find the area of the region enclosed by the graphs of the two functions.

**7.** $y = \sin x,\ y = 1 - x^2$ $\approx 1.670$ **8.** $y = \cos(2x),\ y = x^2 - 2$ $\approx 4.332$

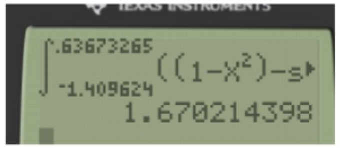





-1.409...

$$\int_{-1.409634}^{.63673265} \left[(1-x^2) - \sin(x)\right] dx$$

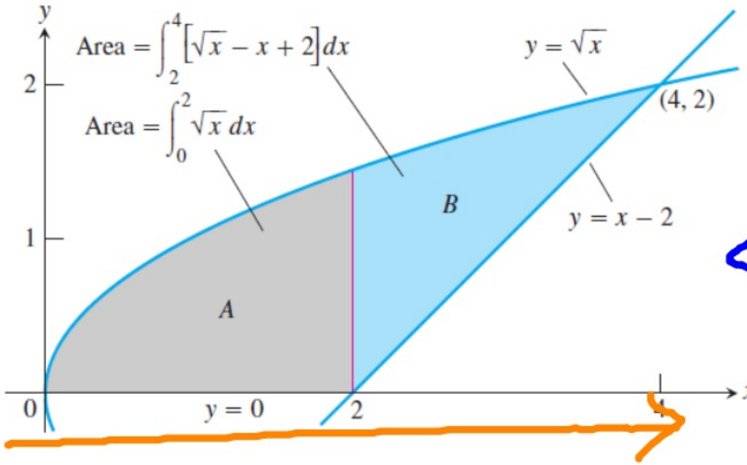


## EXAMPLE 4 Finding Area Using Subregions

Find the area of the region $R$ in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the $x$-axis and the line $y = x - 2$.

**SOLUTION**

The region is shown in Figure 7.10.




$$\int_a^b f(x)\, dx$$

$$\text{Area of } R = \underbrace{\int_0^2 \sqrt{x}\, dx}_{\text{area of } A} + \underbrace{\int_2^4 [\sqrt{x} - (x - 2)]\, dx}_{\text{area of } B}$$

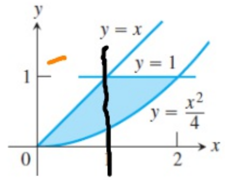
$$= \frac{2}{3}x^{3/2}\Big]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

$$= \frac{10}{3} \text{ units squared}$$

*Now try Exercise 9.*

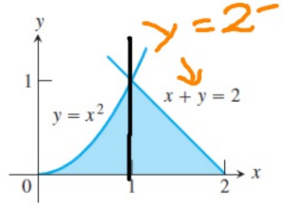
In Exercises 9 and 10, find the area of the shaded region analytically.

**9.** 5/6




$$\int_0^1 \left(x - \frac{x^2}{4}\right)dx + \int_1^2 \left(1 - \frac{x^2}{4}\right)dx$$

**10.** $y = 2-x$ 5/6



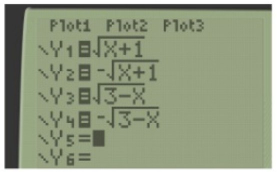

$$\int_0^1 x^2\,dx + \int_1^2 (2-x)\,dx$$

In Exercises 11 and 12, find the area enclosed by the graphs of the two curves by $\boxed{\text{integrating with respect to } y.}$

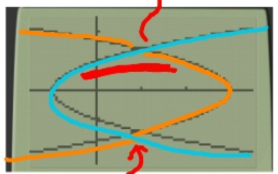
We will do this after going through example 5...

**11.** $y^2 = x + 1$, $y^2 = 3 - x$ ≈7.542  **12.** $y^2 = x + 3$, $y = 2x$ ≈7.146

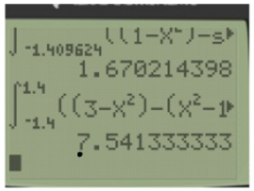
11




1.4 ish



-1.4ish




$y^2 = 3 - x$
$x = 3 - y^2$

$y^2 = x + 1$
$x = y^2 - 1$

$$\int_{-1.4}^{1.4} \left[(3-y^2) - (y^2-1)\right]dy$$

**12.** $y^2 = x + 3,\ y = 2x$

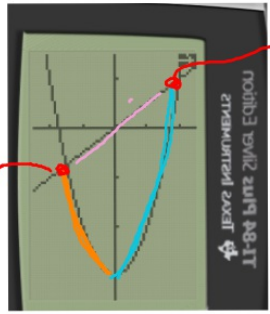
$\approx 7.146$

lower

$(-.75, -1.5)$

upper

$(1, 2)$

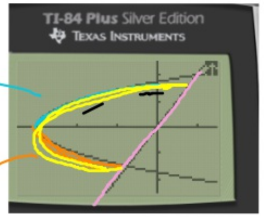



$x = \frac{1}{2}y$

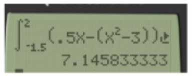
$y^2 - 3 = x$

$y = \sqrt{x+3}$

$y = -\sqrt{x+3}$

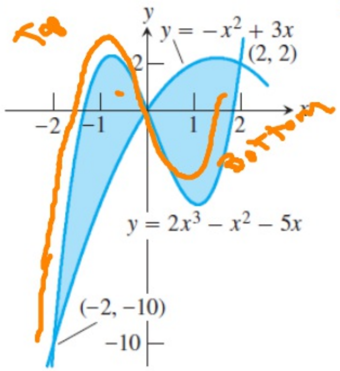



$$\int_{-1.5}^{2} \left[\frac{1}{2}y - (y^2 - 3)\right] dy$$



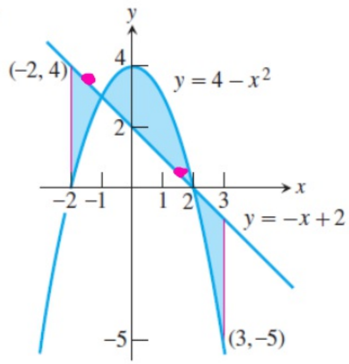
In Exercises 13 and 14, find the total shaded area.

**13.** 16




$$\int_{-2}^{0} \left[(2x^3 - x^2 - 5x) - (-x^2 + 3x)\right] dx + \int_{0}^{2} \left[(-x^2 + 3x) - (2x^3 - x^2 - 5x)\right] dx$$

**14.** $8\frac{1}{6}$




1.8333333 4.5

(14) $$\int_{-2}^{-1} \left[(-x+2) - (4-x^2)\right] dx + \int_{-1}^{2} \left[(4-x^2) - (-x+2)\right] dx + \int_{2}^{3} \left[(-x+2) - (4-x^2)\right] dx = 8\frac{1}{6}$$

In Exercises 15–34, find the area of the regions enclosed by the lines and curves.

**15.** $y = x^2 - 2$ and $y = 2$ $10\frac{2}{3}$

**16.** $y = 2x - x^2$ and $y = -3$ $10\frac{2}{3}$

**17.** $y = 7 - 2x^2$ and $y = x^2 + 4$ $4$

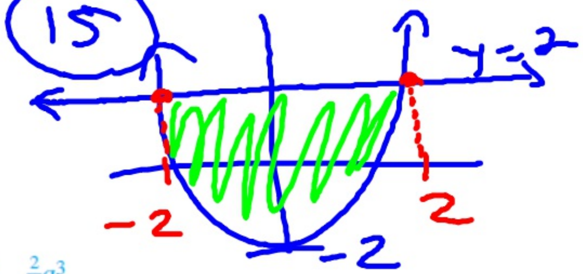
**18.** $y = x^4 - 4x^2 + 4$ and $y = x^2$ $8$

**19.** $y = x\sqrt{a^2 - x^2}$, $a > 0$, and $y = 0$ $\frac{2}{3}a^3$

**20.** $y = \sqrt{|x|}$ and $5y = x + 6$ $1\frac{2}{3}$ (3 points of intersection)
(How many intersection points are there?)

**21.** $y = |x^2 - 4|$ and $y = (x^2/2) + 4$ $21\frac{1}{3}$

15

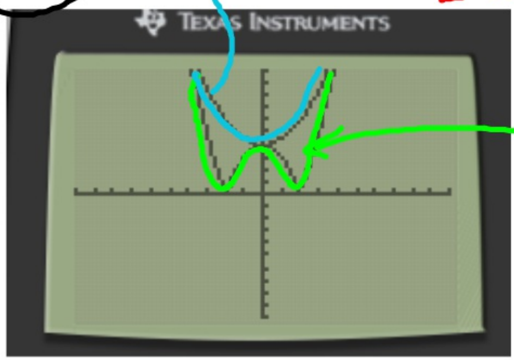



$x^2-2=2$
$x^2=4$
$x=\pm 2$

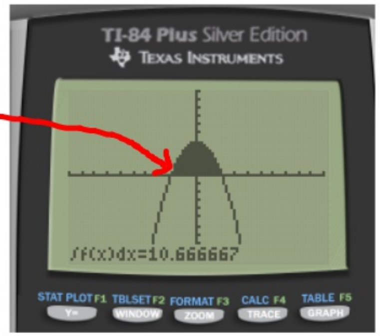
$$\int_{-2}^{2} \left[2-(x^2-2)\right] \quad -x^2+4$$

$(x^2/2)+4$

21




$|x^2-4|$



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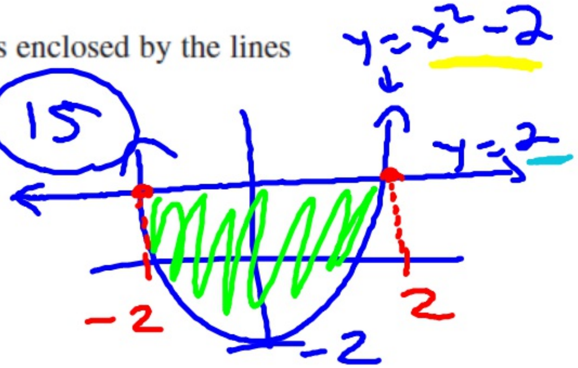
**17.** $y = 7 - 2x^2$ and $y = x^2 + 4$ $4$

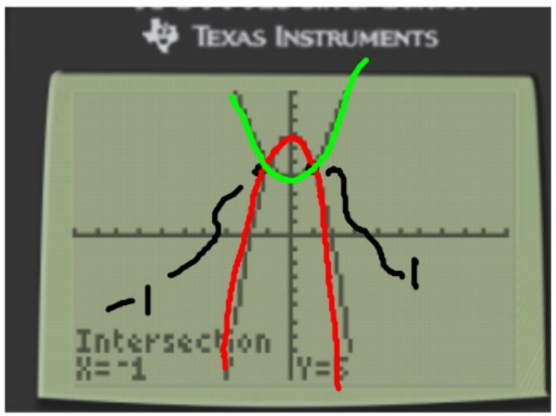
15




$$\int_{-2}^{2} \left[(2) - (x^2 - 2)\right] dx$$

$$\int_{-1}^{1} \left[(7 - 2x^2) - (x^2 + 4)\right] dx$$

17



## EXAMPLE 5 Integrating with Respect to $y$

Find the area of the region in Example 4 by integrating with respect to $y$.

### SOLUTION

We remarked in solving Example 4 that "it appears that no single integral can give the area of $R$," but notice how appearances change when we think of our rectangles being summed over $y$. The interval of integration is $[0, 2]$, and the rectangles run between the same two curves on the entire interval. There is no need to split the region (Figure 7.11).

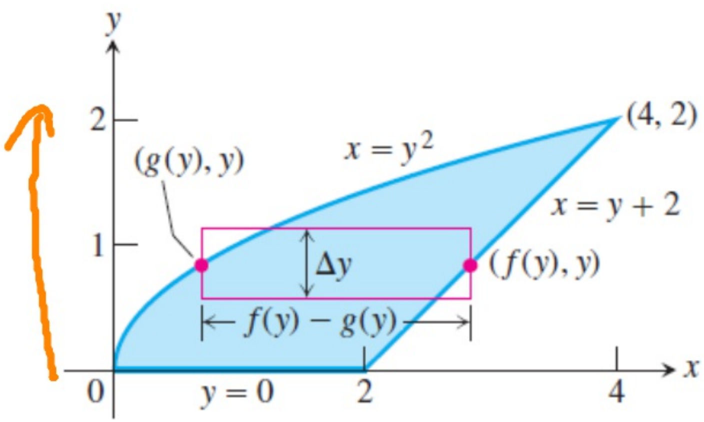
We need to solve for $x$ in terms of $y$ in both equations:

$$y = x - 2 \quad \text{becomes} \quad x = y + 2,$$

$$y = \sqrt{x} \quad \text{becomes} \quad x = y^2, \quad y \geq 0.$$

$$\int_a^b f(y)\,dy$$

*continued*



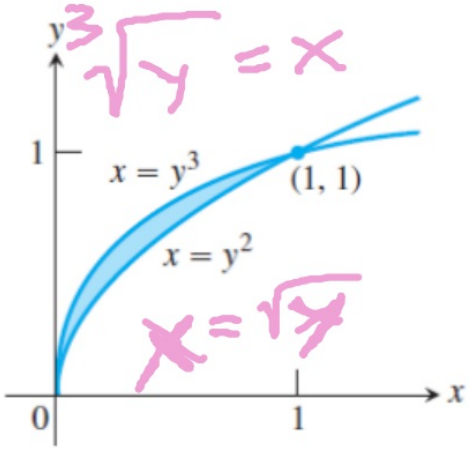

$$\text{Area of } R = \int_0^2 (y + 2 - y^2)\,dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_0^2 = \frac{10}{3} \text{ units squared.}$$

***Now try Exercise 11.***

remember these two...?

The "upper curve" will now be
the rightmost curve, and
the "lower curve," leftmost.

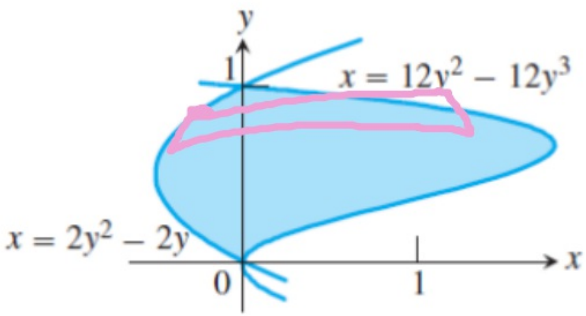
3.




1/12

$$\int_0^1 \left(\sqrt{y} - \sqrt[3]{y}\right) dy$$

4.



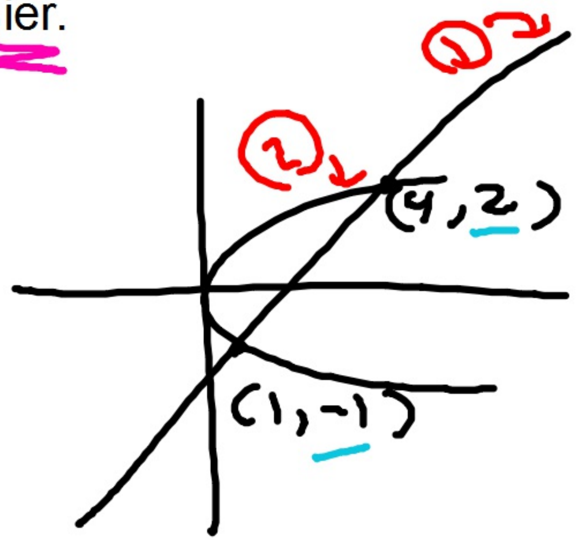

4/3

In Exercises 15–34, find the area of the regions enclosed by the lines and curves.

2 $y = \pm\sqrt{x}$ 1 $y = x - 2$

note: you don't HAVE to always have to integrate along the y-axis, BUT it can make it easier.

22. $x = y^2$ and $x = y + 2$ $4\frac{1}{2}$
23. $y^2 - 4x = 4$ and $4x - y = 16$ $30\frac{3}{8}$
24. $x - y^2 = 0$ and $x + 2y^2 = 3$ 4
25. $x + y^2 = 0$ and $x + 3y^2 = 2$ 8/3 $dx$
26. $4x^2 + y = 4$ and $x^4 - y = 1$ $6\frac{14}{15}$
27. $x + y^2 = 3$ and $4x + y^2 = 0$ 8
28. $y = 2 \sin x$ and $y = \sin 2x$, $0 \le x \le \pi$ 4
29. $y = 8 \cos x$ and $y = \sec^2 x$, $-\pi/3 \le x \le \pi/3$ $6\sqrt{3}$
30. $y = \cos(\pi x/2)$ and $y = 1 - x^2$ $\frac{4}{3} - \frac{4}{\pi} \approx 0.0601$
31. $y = \sin(\pi x/2)$ and $y = x$ $\frac{4 - \pi}{\pi} \approx 0.273$
32. $y = \sec^2 x$, $y = \tan^2 x$, $x = -\pi/4$, $x = \pi/4$ $\frac{\pi}{2}$
33. $x = \tan^2 y$ and $x = -\tan^2 y$, $-\pi/4 \le y \le \pi/4$ $4 - \pi \approx 0.858$
34. $x = 3 \sin y \sqrt{\cos y}$ and $x = 0$, $0 \le y \le \pi/2$ 2




$$\int_{-1}^{2} (y+2) - (y^2)\, dy$$

In Exercises 35–38, use the cylindrical shell method to find the volume of the solid generated by revolving the region bounded by the curves about the $y$-axis.

**35.** $y = x, \quad y = -x/2, \quad x = 2$ $8\pi$

**36.** $y = x^2, \quad y = 2 - x, \quad x = 0, \quad \text{for } x \geq 0$ $5\pi/6$

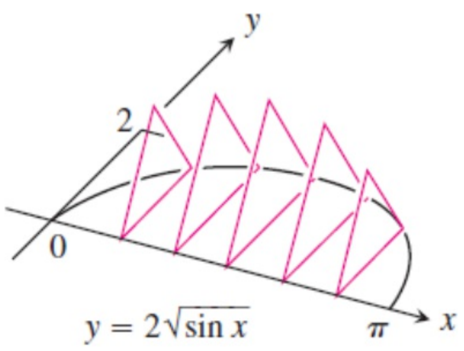
**37.** $y = \sqrt{x}, \quad y = 0, \quad x = 4$ $128\pi/5$

**38.** $y = 2x - 1, \quad y = \sqrt{x}, \quad x = 0$ $7\pi/15$

In Exercises 39–42, find the volume of the solid analytically.

**39.** The base of a solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval $[0, \pi]$ on the $x$-axis. The cross sections perpendicular to the $x$-axis are

**(a)** equilateral triangles with bases running from the $x$-axis to the curve as shown in the figure. $2\sqrt{3}$




**(b)** squares with bases running from the $x$-axis to the curve. $8$

**40.** The solid lies between planes perpendicular to the $x$-axis at $x = -\pi/3$ and $x = \pi/3$. The cross sections perpendicular to the $x$-axis are

**(a)** circular disks with diameters running from the curve $y = \tan x$ to the curve $y = \sec x$. $\pi\sqrt{3} - (\pi^2/6)$

**(b)** squares whose bases run from the curve $y = \tan x$ to the curve $y = \sec x$. $4\sqrt{3} - (2\pi/3)$

**41.** The solid lies between planes perpendicular to the $y$-axis at $y = 0$ and $y = 2$. The cross sections perpendicular to the $y$-axis are circular disks with diameters running from the $y$-axis to the parabola $x = \sqrt{5}y^2$. $8\pi$

**42.** The base of the solid is the disk $x^2 + y^2 \leq 1$. The cross sections by planes perpendicular to the $y$-axis between $y = -1$ and $y = 1$ are isosceles right triangles with one leg in the disk. $8/3$