7.4 Lengths of Curves

$2\frac{4}{4} = \frac{8}{4}$

As the title implies, we can lengths of curves using elements of calculus.

Pythagorean theorem...

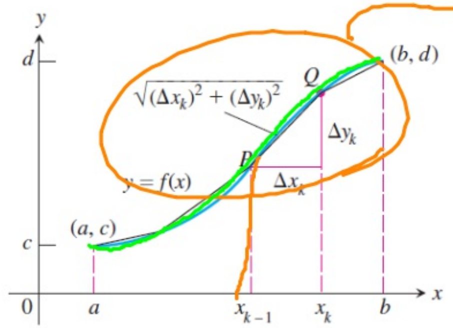


Figure 7.35 The graph of f, approximated by line segments.

$\sqrt{\frac{(\Delta x)^2 + (\Delta y)^2}{(\Delta x)^2}}$

$$\sum \sqrt{\Delta x_k^2 + \Delta y_k^2} = \sum \frac{\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}}{\Delta x_k} \Delta x_k$$

$$= \sum \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} \Delta x_k$$

As each piece gets really small, it turns into integration!

DEFINITION Arc Length: Length of a Smooth Curve

If a smooth curve begins at (a, c) and ends at (b, d), $a < b$, $c < d$, then the **length (arc length) of the curve** is

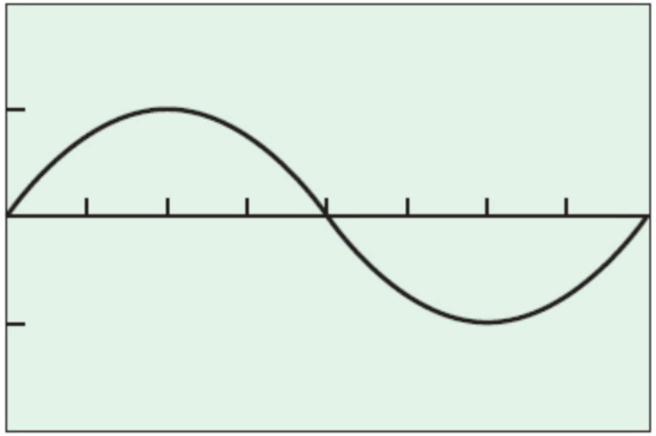
$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{if } y \text{ is a smooth function of } x \text{ on } [a, b];$$

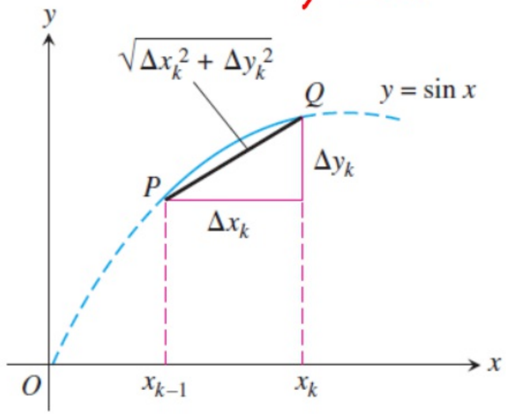
$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \quad \text{if } x \text{ is a smooth function of } y \text{ on } [c, d].$$

EXAMPLE 1 The Length of a Sine Wave

What is the length of the curve $y = \sin x$ from $x = 0$ to $x = 2\pi$?

$y = 0$ $y = 0$





What we'll do is break the curve apart into small segments, then add the segments up.

y'

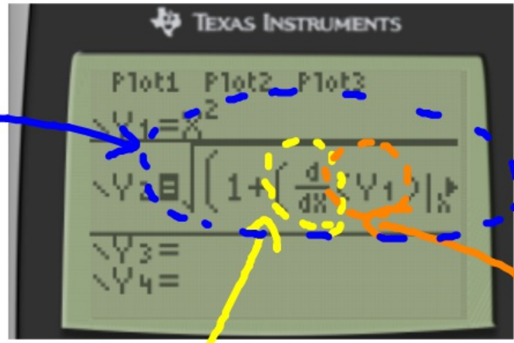
$$\sum \sqrt{1 + (\sin' c_k)^2}\, \Delta x_k,$$

which *is* a Riemann sum.

Now we take the limit as the norms of the subdivisions go to zero and find that the length of one wave of the sine function is

$$\int_0^{2\pi} \sqrt{1 + (\sin' x)^2}\, dx = \int_0^{2\pi} \sqrt{1 + \cos^2 x}\, dx \approx 7.64. \quad \text{Using NINT}$$

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$



In Exercises 1–10,

(a) set up an integral for the length of the curve;

(b) graph the curve to see what it looks like;

(c) use NINT to find the length of the curve.

"lower bound"

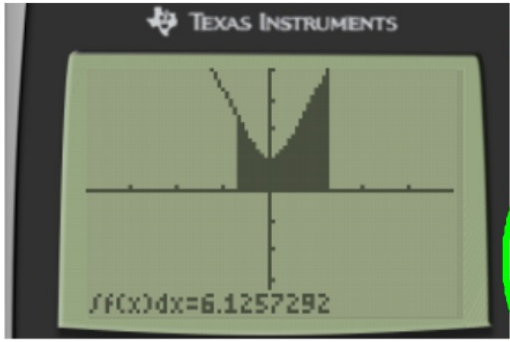
"upper bound"

1. $y = x^2, \quad -1 \le x \le 2$
2. $y = \tan x, \quad -\pi/3 \le x \le 0$
3. $x = \sin y, \quad 0 \le y \le \pi$
4. $x = \sqrt{1 - y^2}, \quad -1/2 \le y \le 1/2$
5. $y^2 + 2y = 2x + 1, \quad$ from $(-1, -1)$ to $(7, 3)$
6. $y = \sin x - x \cos x, \quad 0 \le x \le \pi$
7. $y = \int_0^x \tan t\, dt, \quad 0 \le x \le \pi/6$
8. $x = \int_0^y \sqrt{\sec^2 t - 1}\, dt, \quad -\pi/3 \le y \le \pi/4$
9. $y = \sec x, \quad -\pi/3 \le x \le \pi/3$
10. $y = (e^x + e^{-x})/2, \quad -3 \le x \le 3$

Math → 8:NDeriv

Vars → Y-VARS → 1:Function

2nd → calc → 7:∫f(x)



#7 & 8- what happens when you derive an antiderivative...?

$$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

y'

In Exercises 1–10,

(a) set up an integral for the length of the curve;

(b) graph the curve to see what it looks like;

(c) use NINT to find the length of the curve.

1. $y = x^2, \quad -1 \le x \le 2$

2. $y = \tan x, \quad \overset{a}{-\pi/3} \le \overset{b}{x} \le 0$

3. $x = \sin y, \quad 0 \le y \le \pi$

4. $x = \sqrt{1 - y^2}, \quad -1/2 \le y \le 1/2$

5. $y^2 + 2y = 2x + 1$, from $(-1, -1)$ to $(7, 3)$

6. $y = \sin x - x \cos x, \quad 0 \le x \le \pi$

7. $y = \int_0^x \tan t\, dt, \quad 0 \le x \le \pi/6$

8. $x = \int_0^y \sqrt{\sec^2 t - 1}\, dt, \quad -\pi/3 \le y \le \pi/4$

9. $y = \sec x, \quad -\pi/3 \le x \le \pi/3$

10. $y = (e^x + e^{-x})/2, \quad -3 \le x \le 3$

1 $y = x^2$

$\frac{dy}{dx} = y' = 2x$

$$\int_{-1}^{2} \sqrt{1 + (2x)^2}\,dx$$

... then integrate

5. $y^2 + 2y = 2x + 1$, from $(-1, -1)$ to $(7, 3)$ ... it's easier to do this in terms of "y"...

x_1, y_1 x_2, y_2

$$y^2 + 2y - 1 = 2x$$

$$x = \frac{y^2 + 2y - 1}{2}$$

$$x = \frac{1}{2}y^2 + y - \frac{1}{2}$$

$$x' = y + 1$$

$$\int_{-1}^{3} \sqrt{1 + (y+1)^2}\, dy$$

... then integrate!

7. $y = \int_0^x \tan t \, dt, \quad 0 \le x \le \pi/6$

8. $x = \int_0^y \sqrt{\sec^2 t - 1} \, dt, \quad -\pi/3 \le y \le \pi/4$

9. $y = \sec x, \quad -\pi/3 \le x \le \pi/3$

10. $y = (e^x + e^{-x})/2, \quad -3 \le x \le 3$

7. $y = \int_0^x \tan t \, dt$

$$\frac{dy}{dx} = \tan x$$

$$\int_0^{\pi/6} \sqrt{1 + \tan^2 x}$$

8. $x' = \sqrt{\sec^2 y - 1}$

this goes away once squared...

$$\int_{-\pi/3}^{\pi/4} \sqrt{1 - (\sec^2 y - 1)} \, dy$$

$$= \int_{-\pi/3}^{\pi/4} \sqrt{-\sec^2 y} \, dy$$

7. $y = \int_0^x \tan t \, dt, \quad 0 \le x \le \pi/6$

$y' = \tan x$

$$L = \int_0^{\pi/6} \sqrt{1 + (\tan x)^2} \, dx$$

8. $x = \int_0^y \sqrt{\sec^2 t - 1} \, dt, \quad -\pi/3 \le y \le \pi/4$

$x' = \sqrt{\sec^2 y - 1}$

$$L = \int_{-\pi/3}^{\pi/4} \sqrt{1 + \sec^2 y - 1} \, dy$$

$$= \int_{-\pi/3}^{\pi/4} \sec y \, dy$$

$\frac{1}{\cos y}$

sex

x=y

5. $y^2 + 2y = 2x + 1$, from $(-1, -1)$ to $(7, 3)$

$$x = \frac{y^2 + 2y - 1}{2}$$

$$\int_{-1}^{3} \sqrt{1 + \frac{d}{dx}\left(\frac{y^2+2y-1}{2}\right)^2}\, dy$$

EXAMPLE 2 Applying the Definition

Find the *exact* length of the curve

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1 \quad \text{for} \quad 0 \le x \le 1.$$

SOLUTION

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}x^{1/2} = 2\sqrt{2}\,x^{1/2},$$

which is continuous on [0, 1]. Therefore,

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \\ &= \int_0^1 \sqrt{1 + \left(2\sqrt{2}x^{1/2}\right)^2}\,dx \\ &= \int_0^1 \sqrt{1 + 8x}\,dx \\ &= \frac{2}{3} \cdot \frac{1}{8}(1 + 8x)^{3/2}\Big]_0^1 \\ &= \frac{13}{6}. \end{aligned}$$

Now try Exercise 11.

In Exercises 11–18, find the exact length of the curve analytically by antidifferentiation. You will need to simplify the integrand algebraically before finding an antiderivative.

11. $y = (1/3)(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$ 12

12. $y = x^{3/2}$ from $x = 0$ to $x = 4$ $(80\sqrt{10} - 8)/27$

13. $x = (y^3/3) + 1/(4y)$ from $y = 1$ to $y = 3$
[*Hint:* $1 + (dx/dy)^2$ is a perfect square.] 53/6

14. $x = (y^4/4) + 1/(8y^2)$ from $y = 1$ to $y = 2$
[*Hint:* $1 + (dx/dy)^2$ is a perfect square.] 123/32

15. $x = (y^3/6) + 1/(2y)$ from $y = 1$ to $y = 2$
[*Hint:* $1 + (dx/dy)^2$ is a perfect square.] 17/12

16. $y = (x^3/3) + x^2 + x + 1/(4x + 4)$, $0 \le x \le 2$ 53/6

17. $x = \int_0^y \sqrt{\sec^4 t - 1}\, dt$, $-\pi/4 \le y \le \pi/4$ 2

18. $y = \int_{-2}^x \sqrt{3t^4 - 1}\, dt$, $-2 \le x \le -1$ $7\sqrt{3}/3$

14

$$\int_1^2 \sqrt{1 + (x')^2}$$

17

$$x' = \sqrt{\sec^4 y - 1}$$

$$= \int \sqrt{1 + \sec^4 y - 1}$$

$$= \int_{-\pi/4}^{\pi/4} \sec^2 y$$

$=$

In Exercises 11–18, find the exact length of the curve analytically by antidifferentiation. You will need to simplify the integrand algebraically before finding an antiderivative.

11. $y = (1/3)(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$ 12

12. $y = x^{3/2}$ from $x = 0$ to $x = 4$ $(80\sqrt{10} - 8)/27$

13. $x = (y^3/3) + 1/(4y)$ from $y = 1$ to $y = 3$
[*Hint:* $1 + (dx/dy)^2$ is a perfect square.] 53/6

14. $x = (y^4/4) + 1/(8y^2)$ from $y = 1$ to $y = 2$
[*Hint:* $1 + (dx/dy)^2$ is a perfect square.] 123/32

15. $x = (y^3/6) + 1/(2y)$ from $y = 1$ to $y = 2$
[*Hint:* $1 + (dx/dy)^2$ is a perfect square.] 17/12

16. $y = (x^3/3) + x^2 + x + 1/(4x + 4)$, $0 \le x \le 2$ 53/6

17. $x = \int_0^y \sqrt{\sec^4 t - 1}\, dt$, $-\pi/4 \le y \le \pi/4$ 2

18. $y = \int_{-2}^x \sqrt{3t^4 - 1}\, dt$, $-2 \le x \le -1$ $7\sqrt{3}/3$

(11) $y = \frac{1}{3}(x^2+2)^{3/2}$

$y' = \frac{1}{2}(x^2+2)^{1/2} \cdot 2x$

$f'(x) = x(x^2+2)^{1/2}$

$= \int_0^3 \sqrt{1+(y')^2}\, dy$

12.