8-1 Multiplying and Dividing Rational Expressions

1 Simplify Rational Expressions A ratio of two polynomial expressions such as $\dfrac{1700}{d-33}$ is called a **rational expression**.

Because variables in algebra often represent real numbers, operations with rational numbers and rational expressions are similar. Just as with reducing fractions, to simplify a rational expression, you divide the numerator and denominator by their greatest common factor (GCF).

$$\frac{8}{12} = \frac{2 \cdot \cancel{4}^{1}}{3 \cdot \cancel{4}_{1}} = \frac{2}{3}$$

GCF = 4

$$\frac{x^2 - 4x + 3}{x^2 - 6x + 5} = \frac{(x-3)\cancel{(x-1)}^{1}}{(x-5)\cancel{(x-1)}_{1}} = \frac{(x-3)}{(x-5)}$$

GCF = (x − 1)

Example 1 Simplify a Rational Expression

a. Simplify $\dfrac{5x(x^2 + 4x + 3)}{(x - 6)(x^2 - 9)}$.

$$\dfrac{5x(x^2 + 4x + 3)}{(x - 6)(x^2 - 9)} = \dfrac{5x(x + 3)(x + 1)}{(x - 6)(x + 3)(x - 3)}$$ Factor numerator and denominator.

$$= \dfrac{5x(x + 1)}{(x - 6)(x - 3)} \cdot \dfrac{\cancel{(x + 3)}^{1}}{\cancel{(x + 3)}_{1}}$$ Eliminate common factors.

$$= \dfrac{5x(x + 1)}{(x - 6)(x - 3)}$$ Simplify.

b. Under what conditions is this expression undefined? Remember: you can't divide by zero!

The original factored denominator is $(x - 6)(x + 3)(x - 3)$.
Determine the values that would make the denominator equal to 0.
These values are 6, −3, or 3, so the expression is undefined when $x = 6, -3$ or 3.

Check Your Understanding

◯ = Step-by-Step Solu

Example 1 **Simplify each expression.**

1. $\dfrac{x^2 - 5x - 24}{x^2 - 64} \cdot \dfrac{x + 3}{x + 8}$

2. $\dfrac{c + d}{3c^2 - 3d^2} \cdot \dfrac{1}{3(c - d)}$

$$\dfrac{(x - 8)(x + 3)}{(x + 8)(x - 8)}$$

$$3(c + d)(c - d)$$

$$\frac{2}{4} = \frac{x+1}{x+3} = \frac{1}{3}$$

$.5$

$.\frac{3}{3}$

For what value(s) is $\dfrac{x^2(x^2 - 5x - 14)}{4x(x^2 + 6x + 8)}$ **undefined?**

A $-2, -4$ C $0, -2, -4$

B $-2, 7$ D $0, -2, -4, 7$

Read the Test Item

You want to determine which values of x make the denominator equal to 0.

Solve the Test Item

With $4x$ in the denominator, x cannot equal 0. So, choices A and B can be eliminated. Next, factor the denominator.

$x^2 + 6x + 8 = (x + 2)(x + 4)$, so the denominator is $4x(x + 2)(x + 4)$.

Because the denominator equals 0 when $x = 0, -2,$ and -4, the answer is C.

Example 2   **3. MULTIPLE CHOICE** Identify all values of x for which $\dfrac{x + 7}{x^2 - 3x - 28}$ is undefined. **D**

A $-7, 4$ B $7, 4$ C $4, -7, 7$ D $-4, 7$

$$\frac{x+7}{(x-7)(x+4)}$$

The method for multiplying and dividing fractions also works with rational expressions. Remember that to multiply two fractions, you multiply the numerators and multiply the denominators. To divide two fractions, you multiply by the multiplicative inverse, or the reciprocal, of the divisor.

Multiplication

$$\frac{2}{9} \cdot \frac{15}{4} = \frac{\overset{1}{\cancel{2}} \cdot \overset{1}{\cancel{3}} \cdot 5}{\cancel{3} \cdot 3 \cdot \underset{1}{\cancel{2}} \cdot \underset{1}{\cancel{2}}} = \frac{5}{3 \cdot 2} = \frac{5}{6}$$

Division

$$\frac{3}{5} \div \frac{6}{35} = \frac{3}{5} \cdot \frac{35}{6} = \frac{\overset{1}{\cancel{3}} \cdot \overset{1}{\cancel{5}} \cdot 7}{\cancel{5} \cdot 2 \cdot \underset{1}{\cancel{3}}} = \frac{7}{2}$$

12. **CCSS SENSE-MAKING** The volume of a shipping container in the shape of a rectangular prism can be represented by the polynomial $6x^3 + 11x^2 + 4x$, where the height is x.

   a. Find the length and width of the container. ▭

   b. Find the ratio of the three dimensions of the container when $x = 2$. ▭

   c. Will the ratio of the three dimensions be the same for all values of x? ▭

Volume = $6x^3 + 11x^2 + 4x$
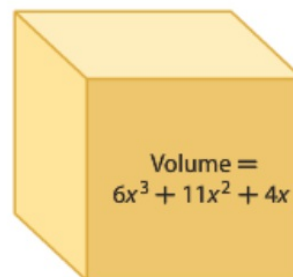
Example 3 Simplify Using −1

Simplify $\dfrac{(4w^2 - 3wy)(w + y)}{(3y - 4w)(5w + y)}$.

Examples 3–6 Simplify each expression.

4. $\dfrac{y^2 + 3y - 40}{25 - y^2}$

$\dfrac{x(a^2 - b^2)}{y(b - a)}$

$3 - 4$

5. $\dfrac{a^2x - b^2x}{by - ay}$ $\dfrac{-x(a + b)}{y}$

$\dfrac{x(a - b)(a + b)}{-y(a - b)}$

7. $\dfrac{12x^3y}{13ab^2} \div \dfrac{36xy^3}{26b}$ $\dfrac{2x^2}{3aby^2}$

b. $\dfrac{18xy^3}{7a^2b^2} \div \dfrac{12x^2y}{35a^2b}$

6. $\dfrac{27x^2y^4}{16yz^3} \cdot \dfrac{8z}{9xy^3}$ $\boxed{}$

7. $\dfrac{12x^3y}{13ab^2} \div \dfrac{36xy^3}{26b}$ $\dfrac{2x^2}{3aby^2}$

7. $\dfrac{12x^3y}{13ab^2} \div \dfrac{36xy^3}{26b} \quad \dfrac{2x^2}{3aby^2}$

$$\dfrac{12x^3y}{13ab^2} \cdot \dfrac{26b}{36xy^3} = \dfrac{2x^2}{3aby^2}$$

Example 5 Polynomials in the Numerator and Denominator

Simplify each expression.

a. $\dfrac{x^2 - 6x - 16}{x^2 - 16x + 64} \cdot \dfrac{x - 8}{x^2 + 5x + 6}$

$\dfrac{x^2 - 6x - 16}{x^2 - 16x + 64} \cdot \dfrac{x - 8}{x^2 + 5x + 6} = \dfrac{(x - 8)(x + 2)}{(x - 8)(x - 8)} \cdot \dfrac{x - 8}{(x + 3)(x + 2)}$ **Factor.**

$= \dfrac{(x - 8)(x + 2)}{(x - 8)(x - 8)} \cdot \dfrac{x - 8}{(x + 3)(x + 2)}$ **Eliminate common factors.**

$= \dfrac{1}{x + 3}$ **Simplify.**

b. $\dfrac{x^2 - 16}{12y + 36} \div \dfrac{x^2 - 12x + 32}{y^2 - 3y - 18}$

$\dfrac{x^2 - 16}{12y + 36} \div \dfrac{x^2 - 12x + 32}{y^2 - 3y - 18} = \dfrac{x^2 - 16}{12y + 36} \cdot \dfrac{y^2 - 3y - 18}{x^2 - 12x + 32}$ **Multiply by reciprocal.**

$= \dfrac{(x + 4)(x - 4)}{12(y + 3)} \cdot \dfrac{(y - 6)(y + 3)}{(x - 4)(x - 8)}$ **Factor.**

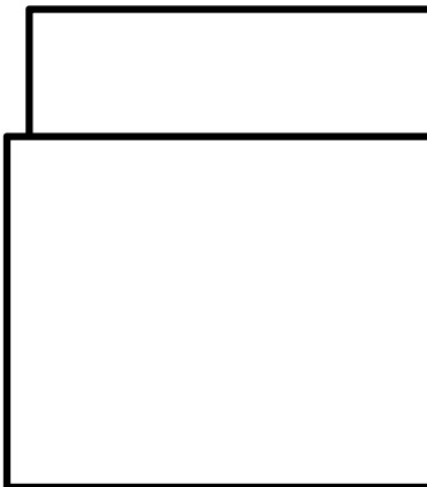$= \dfrac{(x + 4)(x - 4)}{12(y + 3)} \cdot \dfrac{(y - 6)(y + 3)}{(x - 4)(x - 8)}$ **Eliminate common factors.**

$= \dfrac{(x + 4)(y - 6)}{12(x - 8)}$ **Simplify.**

8. $\dfrac{x^2 - 4x - 21}{x^2 - 6x + 8} \cdot \dfrac{x - 4}{x^2 - 2x - 35}$ **$\dfrac{x + 3}{(x - 2)(x + 5)}$**

9. $\dfrac{a^2 - b^2}{3a^2 - 6a + 3} \div \dfrac{4a + 4b}{a^2 - 1}$ **$\dfrac{(a - b)(a + 1)}{12(a - 1)}$**

$$\text{(11)} \quad \frac{4x}{x+6} \div \frac{x^2-3x}{x^2+3x-18} \qquad \leftarrow \text{flip, factor!}$$

$$\frac{4x}{x+6} \cdot \frac{(x+6)(x-3)}{x(x-3)} \leftarrow$$

10. $\dfrac{\dfrac{a^3b^3}{xy^4}}{\dfrac{a^2b}{x^2y}}$ ▢

11. $\dfrac{\dfrac{4x}{x+6}}{\dfrac{x^2-3x}{x^2+3x-18}}$ **4**

Example 1 Simplify each expression.

13. $\dfrac{x(x-3)(x+6)}{x^2+x-12}$ $\dfrac{x(x+6)}{x+4}$

14. $\dfrac{y^2(y^2+3y+2)}{2y(y-4)(y+2)}$ $\dfrac{y(y+1)}{2(y-4)}$

15. $\dfrac{(x^2-9)(x^2-z^2)}{4(x+z)(x-3)}$ $\dfrac{(x+3)(x-z)}{4}$

16. $\dfrac{(x^2-16x+64)(x+2)}{(x^2-64)(x^2-6x-16)}$ $\dfrac{1}{x+8}$

17. $\dfrac{x^2(x+2)(x-4)}{6x(x^2+x-20)}$ $\dfrac{x(x+2)}{6(x+5)}$

18. $\dfrac{3y(y-8)(y^2+2y-24)}{15y^2(y^2-12y+32)}$ $\dfrac{(y+6)}{5y}$

Example 2 **19. MULTIPLE CHOICE** Identify all values of x for which $\dfrac{(x-3)(x+6)}{(x^2-7x+12)(x^2-36)}$ is undefined. **J**

 F $3, -6$ **G** $4, 6$ **H** $-6, 6$ **J** $-6, 3, 4, 6$

Example 3 Simplify each expression.

20. $\dfrac{x^2-5x-14}{28+3x-x^2}$ $\dfrac{x+2}{x+4}$

21. $\dfrac{x^3-9x^2}{x^2-3x-54}$ $\dfrac{x^2}{x+6}$

22. $\dfrac{(x-4)(x^2+2x-48)}{(36-x^2)(x^2+4x-32)}$ $-\dfrac{1}{x+6}$

23. $\dfrac{16-c^2}{c^2+c-20}$ $-\dfrac{c+4}{c+5}$

24. GEOMETRY The cylinder at the right has a volume of $(x+3)(x^2-3x-18)\pi$ cubic centimeters. Find the height of the cylinder. **$x-6$ cm**

2x + 6 cm
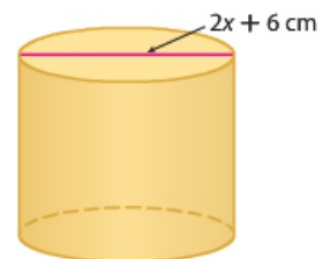
Examples 4–6 Simplify each expression.

25. $\dfrac{3ac^3f^3}{8a^2bcf^4} \cdot \dfrac{12ab^2c}{18ab^3c^2f}$ $\dfrac{c}{4ab^2f^2}$

26. $\dfrac{14xy^2z^3}{21w^4x^2yz} \cdot \dfrac{7wxyz}{12w^2y^3z}$ $\dfrac{7z^2}{18w^5y}$

27. $\dfrac{64a^2b^5}{35b^2c^3f^4} \div \dfrac{12a^4b^3c}{70abcf^2}$ $\dfrac{32b}{3ac^3f^2}$

28. $\dfrac{9x^2yz}{5z^4} \div \dfrac{12x^4y^2}{50xy^4z^2}$ $\dfrac{15y^3}{2xz}$

29. $\dfrac{15a^2b^2}{21ac} \cdot \dfrac{14a^4c^2}{6ab^3}$ $\dfrac{5a^4c}{3b}$

30. $\dfrac{14c^2f^5}{9a^2} \div \dfrac{35cf^4}{18ab^3}$ $\dfrac{4b^3cf}{5a}$

31. $\dfrac{y^2+8y+15}{y-6} \cdot \dfrac{y^2-9y+18}{y^2-9}$ $y+5$

32. $\dfrac{c^2-6c-16}{c^2-d^2} \div \dfrac{c^2-8c}{c+d}$ $\dfrac{c+2}{c(c-d)}$

33. $\dfrac{x^2+9x+20}{8x+16} \cdot \dfrac{4x^2+16x+16}{x^2-25}$ $\dfrac{(x+4)(x+2)}{2(x-5)}$

34. $\dfrac{3a^2+6a+3}{a^2-3a-10} \div \dfrac{12a^2-12}{a^2-4}$ $\dfrac{(a+1)(a-2)}{4(a-5)(a-1)}$

35. $\dfrac{\dfrac{x^2-9}{6x-12}}{\dfrac{x^2+10x+21}{x^2-x-2}}$ $\dfrac{(x-3)(x+1)}{6(x+7)}$

36. $\dfrac{\dfrac{y-x}{z^3}}{\dfrac{x-y}{6z^2}}$ $-\dfrac{6}{z}$

37. $\dfrac{\dfrac{a^2-b^2}{b^3}}{\dfrac{b^2-ab}{a^2}}$ $\dfrac{-a^2(a+b)}{b^4}$

38. $\dfrac{\dfrac{x-y}{a+b}}{\dfrac{x^2-y^2}{b^2-a^2}}$ $\dfrac{b-a}{x+y}$

39. **CCSS REASONING** At the end of her high school soccer career, Ashley had made 33 goals out of 121 attempts.

    a. Write a ratio to represent the ratio of the number of goals made to goals attempted by Ashley at the end of her high school career. $\dfrac{33}{121}$

    b. Suppose Ashley attempted a goals and made m goals during her first year at college. Write a rational expression to represent the ratio of the number of career goals made to the number of career goals attempted at the end of her first year in college. $\dfrac{33+m}{121+a}$