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## 8-5 Variation Functions

"Direct" "k"

"x varies directly" $\frac{x}{y} = \frac{1}{2} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10} = .5$

"As x increases, y increases."

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"Inverse"

"x varies inversely to y" $xy = 1 \cdot 12 = 2 \cdot 6 = 3 \cdot 4 = 12 \cdot 1$

"As x increases, y decreases."

$k = 12$

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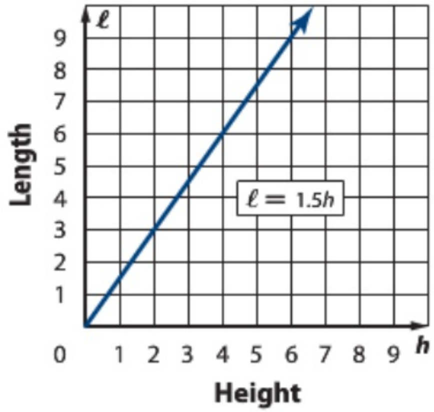
"x varies jointly to y and z..."

$$k = \frac{x}{yz}$$

## 1 Direct Variation and Joint Variation

The relationship given by $\ell = 1.5h$ is an example of direct variation. A **direct variation** can be expressed in the form $y = kx$. In this equation, $k$ is called the **constant of variation**.

Notice that the graph of $\ell = 1.5h$ is a straight line through the origin. A direct variation is a special case of an equation written in slope-intercept form, $y = mx + b$. When $m = k$ and $b = 0$, $y = mx + b$ becomes $y = kx$. So the slope of a direct variation equation is its constant of variation.




To express a direct variation, we say that $y$ varies directly as $x$. In other words, as $x$ increases, $y$ increases or decreases at a constant rate.

### KeyConcept Direct Variation

| | |
|---|---|
| **Words** | $y$ varies directly as $x$ if there is some nonzero constant $k$ such that $y = kx$. $k$ is called the *constant of variation.* |
| **Example** | If $y = 3x$ and $x = 7$, then $y = 3(7)$ or 21. |

## KeyConcept Direct Variation

**Words** $y$ varies directly as $x$ if there is some nonzero constant $k$ such that $y = kx$. $k$ is called the *constant of variation.*

**Example** If $y = 3x$ and $x = 7$, then $y = 3(7)$ or 21.

If you know that $y$ varies directly as $x$ and one set of values, you can use a proportion to find the other set of corresponding values.

$$y_1 = kx_1 \quad \text{and} \quad y_2 = kx_2$$

$$\frac{y_1}{x_1} = k \qquad \frac{y_2}{x_2} = k \qquad \text{Therefore, } \frac{y_1}{x_1} = \frac{y_2}{x_2}.$$

Using the properties of equality, you can find many other proportions that relate these same $x$- and $y$-values.

## Example 1 Direct Variation

**If $y$ varies directly as $x$ and $y = 15$ when $x = -5$, find $y$ when $x = 7$.**

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**Examples 1–3** **1.** If $y$ varies directly as $x$ and $y = 12$ when $x = 8$, find $y$ when $x = 14$. **21**

Another type of variation is joint variation. **Joint variation** occurs when one quantity varies directly as the product of two or more other quantities.

**StudyTip**

**Joint Variation** Some mathematicians consider joint variation a special type of combined variation.

**KeyConcept Joint Variation**

**Words** $y$ varies jointly as $x$ and $z$ if there is some nonzero constant $k$ such that $y = kxz$.

**Example** If $y = 5xz$, $x = 6$, and $z = -2$, then $y = 5(6)(-2)$ or $-60$.

If you know that $y$ varies jointly as $x$ and $z$ and one set of values, you can use a proportion to find the other set of corresponding values.

$$y_1 = kx_1z_1 \qquad \text{and} \qquad y_2 = kx_2z_2$$

$$\frac{y_1}{x_1z_1} = k \qquad \frac{y_2}{x_2z_2} = k \qquad \text{Therefore, } \frac{y_1}{x_1z_1} = \frac{y_2}{x_2z_2}.$$

## KeyConcept Joint Variation

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### Example 2 Joint Variation

**Suppose $y$ varies jointly as $x$ and $z$. Find $y$ when $x = 9$ and $z = 2$, if $y = 20$ when $z = 3$ and $x = 5$.**

Use a proportion that relates the values.

$$\frac{y_1}{x_1 z_1} = \frac{y_2}{x_2 z_2}$$ Joint variation

$$\frac{20}{5(3)} = \frac{y_2}{9(2)}$$ $y_1 = 20$, $x_1 = 5$, $z_1 = 3$, $x_2 = 9$, and $z_2 = 2$

$$20(9)(2) = 5(3)(y_2)$$ Cross multiply.

$$360 = 15y_2$$ Simplify.

$$24 = y_2$$ Divide each side by 15.

### GuidedPractice

## KeyConcept Joint Variation

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| **Words** | $y$ varies jointly as $x$ and $z$ if there is some nonzero constant $k$ such that $y = kxz$. |
| **Example** | If $y = 5xz$, $x = 6$, and $z = -2$, then $y = 5(6)(-2)$ or $-60$. |

**2.** Suppose $y$ varies jointly as $x$ and $z$. Find $y$ when $x = 9$ and $z = -3$, if $y = -50$ when $z$ is 5 and $x$ is $-10$. **−27**

$$\frac{y}{xz} = \frac{y}{(9)(-3)} = \frac{-50}{(-10)(5)}$$

$$\frac{y}{-27} = \frac{-50}{-50}$$

## 2 Inverse Variation and Combined Variation

**Inverse Variation and Combined Variation** Another type of variation is inverse variation. If two quantities $x$ and $y$ show **inverse variation**, their product is equal to a constant $k$.

Inverse variation is often described as one quantity increasing while the other quantity is decreasing. For example, speed and time for a fixed distance vary inversely with each other; the faster you go, the less time it takes you to get there.

### KeyConcept Inverse Variation

| | |
|---|---|
| **Words** | $y$ varies inversely as $x$ if there is some nonzero constant $k$ such that $xy = k$ or $y = \frac{k}{x}$, where $x \neq 0$ and $y \neq 0$. |
| **Example** | If $xy = 2$, and $x = 6$, then $y = \frac{2}{6}$ or $\frac{1}{3}$. |

## KeyConcept Inverse Variation

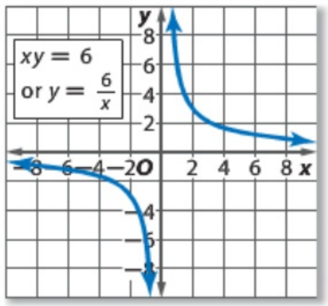
**Words** $y$ varies inversely as $x$ if there is some nonzero constant $k$ such that $xy = k$ or $y = \frac{k}{x}$, where $x \neq 0$ and $y \neq 0$.

**Example** If $xy = 2$, and $x = 6$, then $y = \frac{2}{6}$ or $\frac{1}{3}$.

### StudyTip

**Direct and Inverse Variation** You can identify the type of variation by looking at a table of values for $x$ and $y$. If the quotient $\frac{y}{x}$ has a constant value, $y$ varies directly as $x$. If the product $xy$ has a constant value, $y$ varies inversely as $x$.

Suppose $y$ varies inversely as $x$ such that $xy = 6$ or $y = \frac{6}{x}$. The graph of this equation is shown at the right. Since $k$ is a positive value, as the values of $x$ increase, the values of $y$ decrease.




Notice that the graph of an inverse variation is a reciprocal function.

## KeyConcept Inverse Variation

| | |
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| Words | $y$ varies inversely as $x$ if there is some nonzero constant $k$ such that $xy = k$ or $y = \frac{k}{x}$, where $x \neq 0$ and $y \neq 0$. |
| Example | If $xy = 2$, and $x = 6$, then $y = \frac{2}{6}$ or $\frac{1}{3}$. |

## Example 3 Inverse Variation

PT

**If $a$ varies inversely as $b$ and $a = 28$ when $b = -2$, find $a$ when $b = -10$.**

$xy = (-18)(16) = x(9)$

**3.** If $y$ varies inversely as $x$ and $y = -18$ when $x = 16$, find $x$ when $y = 9$. **−32**

## KeyConcept Inverse Variation

**Words** $y$ varies inversely as $x$ if there is some nonzero constant $k$ such that $xy = k$ or $y = \frac{k}{x}$, where $x \neq 0$ and $y \neq 0$.

**Example** If $xy = 2$, and $x = 6$, then $y = \frac{2}{6}$ or $\frac{1}{3}$.

## Real-World Example 4 Write and Solve an Inverse Variation

**MUSIC The length of a violin string varies inversely as the frequency of its vibrations. A violin string 10 inches long vibrates at a frequency of 512 cycles per second. Find the frequency of an 8-inch violin string.**

**Example 4**

4. **TRAVEL** A map of Illinois is scaled so that 2 inches represents 15 miles. How far apart are Chicago and Rockford if they are 12 inches apart on the map?

Another type of variation is combined variation. **Combined variation** occurs when one quantity varies directly and/or inversely as two or more other quantities.

If you know that $y$ varies directly as $x$, $y$ varies inversely as $z$, and one set of values, you can use a proportion to find the other set of corresponding values.

$$y_1 = \frac{kx_1}{z_1} \qquad \text{and} \qquad y_2 = \frac{kx_2}{z_2}$$

$$\frac{y_1 z_1}{x_1} = k \qquad\qquad \frac{y_2 z_2}{x_2} = k$$

Therefore, $\frac{y_1 z_1}{x_1} = \frac{y_2 z_2}{x_2}$.

## Example 5 Combined Variation

**Suppose $f$ varies directly as $g$, and $f$ varies inversely as $h$. Find $g$ when $f = 18$ and $h = -3$, if $g = 24$ when $h = 2$ and $f = 6$.**

**Example 5**

**5.** Suppose $a$ varies directly as $b$, and $a$ varies inversely as $c$. Find $b$ when $a = 8$ and $c = -3$, if $b = 16$ when $c = 2$ and $a = 4$.

**6.** Suppose $d$ varies directly as $f$, and $d$ varies inversely as $g$. Find $g$ when $d = 6$ and $f = -7$, if $g = 12$ when $d = 9$ and $f = 3$.

## Practice and Problem Solving

Extra Practice is on page R8.

**Example 1** **If $x$ varies directly as $y$, find $x$ when $y = 8$.**

7. $x = 6$ when $y = 32$ **1.5**

8. $x = 11$ when $y = -3$ $-\frac{88}{3}$

9. $x = 14$ when $y = -2$ **−56**

10. $x = -4$ when $y = 10$ **−3.2**

11. **MOON** Astronaut Neil Armstrong, the first man on the Moon, weighed 360 pounds on Earth with all his equipment on, but weighed only 60 pounds on the Moon. Write an equation that relates weight on the Moon $m$ with weight on Earth $w$. $m = \frac{1}{6}w$

**Example 2** **If $a$ varies jointly as $b$ and $c$, find $a$ when $b = 4$ and $c = -3$.**

12. $a = -96$ when $b = 3$ and $c = -8$ **−48**

13. $a = -60$ when $b = -5$ and $c = 4$ **−36**

14. $a = -108$ when $b = 2$ and $c = 9$ **72**

15. $a = 24$ when $b = 8$ and $c = 12$ **−3**

16. CCSS **MODELING** According to the A.C. Nielsen Company, the average American watches 4 hours of television a day.

   a. Write an equation to represent the average number of hours spent watching television by $m$ household members during a period of $d$ days. $t = 4md$

   b. Assume that members of your household watch the same amount of television each day as the average American. How many hours of television would the members of your household watch in a week? **Sample answer for four household members: 112 hours**

**Example 3** **If $f$ varies inversely as $g$, find $f$ when $g = -6$.**

**17.** $f = 15$ when $g = 9$ **$-22.5$**

**18.** $f = 4$ when $g = 28$ **$-\frac{56}{3}$**

**19.** $f = -12$ when $g = 19$ **38**

**20.** $f = 0.6$ when $g = -21$ **2.1**

**21.** **COMMUNITY SERVICE** Every year students at West High School collect canned goods for a local food pantry. They plan to distribute flyers to homes in the community asking for donations. Last year, 12 students were able to distribute 1000 flyers in four hours.

**a.** Write an equation that relates the number of students $s$ to the amount of time $t$ it takes to distribute 1000 flyers. **$s = \frac{48}{t}$**

**b.** How long would it take 15 students to hand out the same number of flyers this year? **3.2 hours**

**Example 4** **22.** **BIRDS** When a group of snow geese migrate, the distance that they fly varies directly with the amount of time they are in the air.

**a.** A group of snow geese migrated 375 miles in 7.5 hours. Write a direct variation equation that represents this situation. **$d = 50t$**

**b.** Every year, geese migrate 3000 miles from their winter home in the southwest United States to their summer home in the Canadian Arctic. Estimate the number of hours of flying time that it takes for the geese to migrate. **60 hours**

**Example 5** **23.** Suppose $a$ varies directly as $b$, and $a$ varies inversely as $c$. Find $b$ when $a = 5$ and $c = -4$, if $b = 12$ when $c = 3$ and $a = 8$. **$-10$**

**24.** Suppose $x$ varies directly as $y$, and $x$ varies inversely as $z$. Find $z$ when $x = 10$ and $y = -7$, if $z = 20$ when $x = 6$ and $y = 14$. **$-6$**