8-6 Solving Rational Equations And Inequalities

● A gaming club charges $20 per month for membership. Members also have to pay $5 each time they visit the club. If a member visits the club x times in one month, then the charge for that month will be $20 + 5x$. The actual cost per visit will be $\frac{20 + 5x}{x}$. To determine how many visits are needed for the cost per visit to be $6, you would need to solve the equation $\frac{20 + 5x}{x} = 6$.

Example 1 Solve a Rational Equation

Solve $\dfrac{4}{x+3} + \dfrac{5}{6} = \dfrac{23}{18}$. Check your solution.

The LCD for the terms is $18(x+3)$.

$$\dfrac{4}{x+3} + \dfrac{5}{6} = \dfrac{23}{18} \qquad \text{Original equation}$$

$$18(x+3)\left(\dfrac{4}{x+3}\right) + 18(x+3)\left(\dfrac{5}{6}\right) = 18(x+3)\dfrac{23}{18} \qquad \text{Multiply by LCD.}$$

$$\overset{1}{\cancel{18(x+3)}}\left(\dfrac{4}{\underset{1}{\cancel{x+3}}}\right) + \overset{3}{\cancel{18}}(x+3)\left(\dfrac{5}{\underset{1}{\cancel{6}}}\right) = \overset{1}{\cancel{18}}(x+3)\left(\dfrac{23}{\underset{1}{\cancel{18}}}\right) \qquad \text{Divide common factors.}$$

$$72 + 15x + 45 = 23x + 69 \qquad \text{Multiply.}$$

$$15x + 117 = 23x + 69 \qquad \text{Simplify.}$$

$$48 = 8x \qquad \text{Subtract } 15x \text{ and } 69.$$

$$6 = x \qquad \text{Divide.}$$

CHECK $\dfrac{4}{x+3} + \dfrac{5}{6} = \dfrac{23}{18}$ $\qquad$ Original equation

$\dfrac{4}{6+3} + \dfrac{5}{6} \overset{?}{=} \dfrac{23}{18}$ $\qquad$ $x = 6$

$\dfrac{4}{9} + \dfrac{5}{6} \overset{?}{=} \dfrac{23}{18}$ $\qquad$ Simplify.

$\dfrac{8}{18} + \dfrac{15}{18} \overset{?}{=} \dfrac{23}{18}$ $\qquad$ Simplify.

$\dfrac{23}{18} = \dfrac{23}{18}$ ✓ $\qquad$ Add.

Examples 1–2 Solve each equation. Check your solution.

1. $\dfrac{4}{7} + \dfrac{3}{x-3} = \dfrac{53}{56}$ **11**

2. $\dfrac{7}{3\,\square} - \dfrac{3}{\square-5} = \dfrac{19}{12}$ $\square$

3. $\dfrac{10}{2x+1} + \dfrac{4}{3} = 2$ **7**

4. $\dfrac{1}{4} - \dfrac{5}{y+3} = \dfrac{23}{12}$ $\square$

LCD:
$(2x+1)(3)$

LCD: $56(x-3)$

① $\dfrac{4\,\overset{8}{(56)(x-3)}}{7} + \dfrac{3\,(56)(x-3)}{x-3} = \dfrac{53\,(56)}{56}(x-3)$

$32(x-3) + 3(56) = 53(x-3)$

$32x - 94 + 168 = 53x - 159$

... solve for x :)

Example 2 Solve a Rational Equation

Solve $\dfrac{2x}{x+5} - \dfrac{x^2 - x - 10}{x^2 + 8x + 15} = \dfrac{3}{x+3}$. Check your solution.

The LCD for the terms is $(x+3)(x+5)$.

$$\dfrac{2x}{x+5} - \dfrac{x^2 - x - 10}{x^2 + 8x + 15} = \dfrac{3}{x+3}$$ Original equation

$$\dfrac{(x+3)(x+5)(2x)}{x+5} - \dfrac{(x+3)(x+5)(x^2 - x - 10)}{x^2 + 8x + 15} = \dfrac{(x+3)(x+5)3}{x+3}$$ Multiply by LCD.

Divide common factors.

$$\dfrac{(x+3)(\cancel{x+5})(2x)}{\cancel{x+5}} - \dfrac{(\cancel{x+3})(\cancel{x+5})(x^2 - x - 10)}{\cancel{x^2 + 8x + 15}} = \dfrac{(x+5)(\cancel{x+3})3}{\cancel{x+3}}$$

$$(x+3)(2x) - (x^2 - x - 10) = 3(x+5)$$ Simplify.

$$2x^2 + 6x - x^2 + x + 10 = 3x + 15$$ Distribute.

$$x^2 + 7x + 10 = 3x + 15$$ Simplify.

$$x^2 + 4x - 5 = 0$$ Subtract $3x + 15$.

$$(x+5)(x-1) = 0$$ Factor.

$$x + 5 = 0 \quad \text{or} \quad x - 1 = 0$$ Zero Product Property

$$x = -5 \qquad\qquad x = 1$$

CHECK Try $x = -5$.

$$\frac{2x}{x+5} - \frac{x^2 - x - 10}{x^2 + 8x + 15} = \frac{3}{x+3}$$

$$\frac{2(-5)}{-5+5} - \frac{(-5)^2 - (-5) - 10}{(-5)^2 + 8(-5) + 15} \overset{?}{=} \frac{3}{-5+3}$$

$$\frac{-10}{0} - \frac{25 + 5 - 10}{25 - 40 + 15} \neq -\frac{3}{2} \quad \text{✗}$$

Try $x = 1$.

$$\frac{2x}{x+5} - \frac{x^2 - x - 10}{x^2 + 8x + 15} = \frac{3}{x+3}$$

$$\frac{2(1)}{1+5} - \frac{1^2 - 1 - 10}{1^2 + 8(1) + 15} \overset{?}{=} \frac{3}{1+3}$$

$$\frac{2}{6} - \frac{-10}{24} \overset{?}{=} \frac{3}{4}$$

$$\frac{8}{24} + \frac{10}{24} \overset{?}{=} \frac{3}{4}$$

$$\frac{3}{4} = \frac{3}{4} \quad \text{✓}$$

When solving a rational equation, any possible solution that results in a zero in the denominator must be excluded from your list of solutions.

Since $x = -5$ results in a zero in the denominator, it is extraneous. Eliminate -5 from the list of solutions. The solution is 1.

5. $\dfrac{8}{x-5} - \dfrac{9}{x-4} = \dfrac{5}{x^2-9x+20}$ **8**

6. $\dfrac{14}{x+3} + \dfrac{10}{x-2} = \dfrac{122}{x^2+x-6}$ ☐

7. $\dfrac{14}{x-8} - \dfrac{5}{x-6} = \dfrac{82}{x^2-14x+48}$ **14**

8. $\dfrac{5}{x+2} - \dfrac{3}{x-2} = \dfrac{12}{x^2-4}$ ☐

$(x-5)(x-4)$

$\boxed{5}$ $\dfrac{8}{x-5} - \dfrac{9}{x-4} = \dfrac{5}{(x-5)(x-4)}$
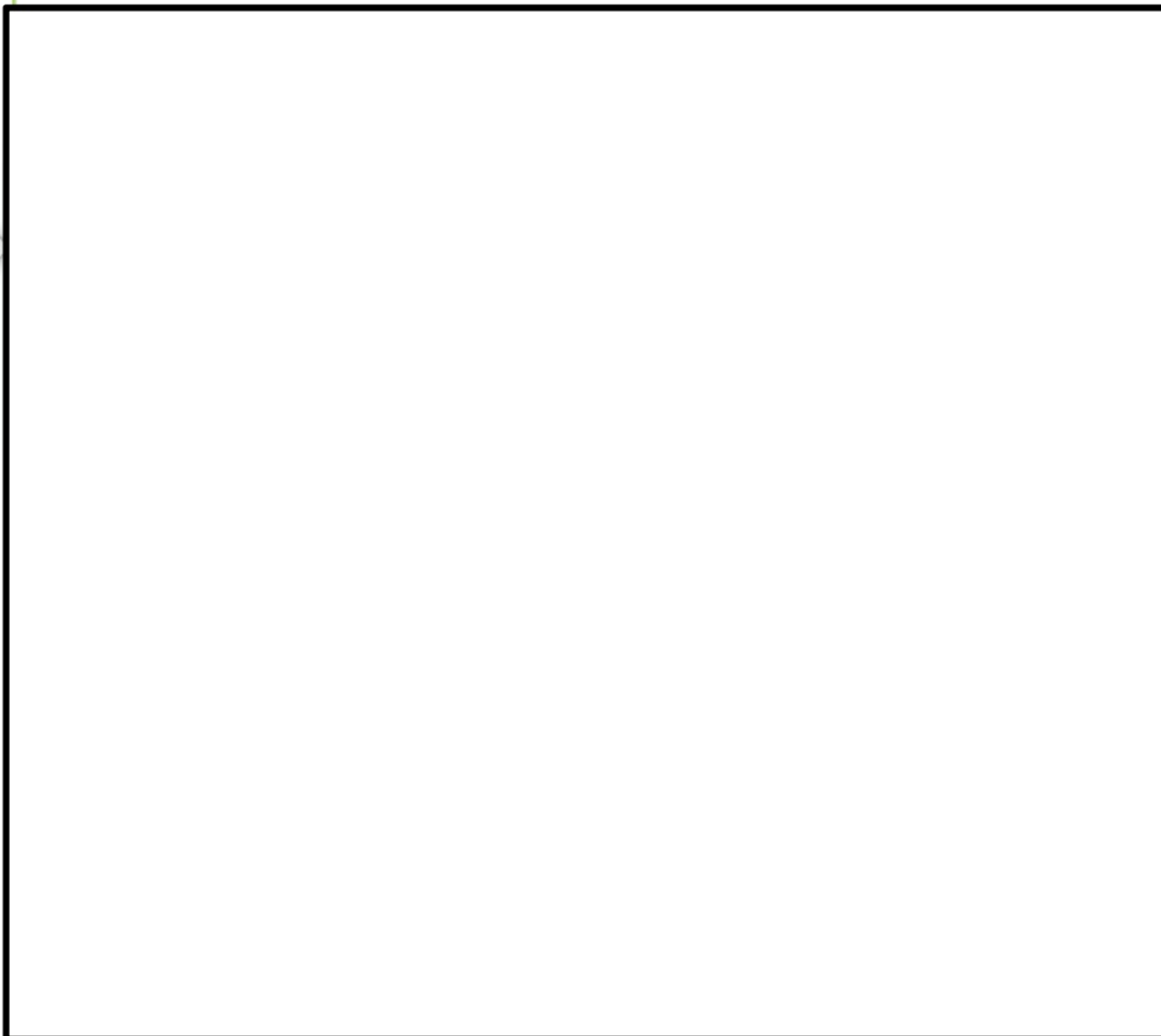
$8(x-4) - 9(x-5) = 5$

$8x - 32 - 9x + 45 = 5$

$-x + 13 = 5$

$x = 13-5$

CHEMISTRY Mia adds a 70% acid solution to 12 milliliters of a solution that is 15% acid. How much of the 70% acid solution should be added to create a solution that is 60% acid?

Example 3

9. **CCSS** **STRUCTURE** Sara has 10 pounds of dried fruit selling for $6.25 per pound. She wants to know how many pounds of mixed nuts selling for $4.50 per pound she needs to make a trail mix selling for $5 per pound.

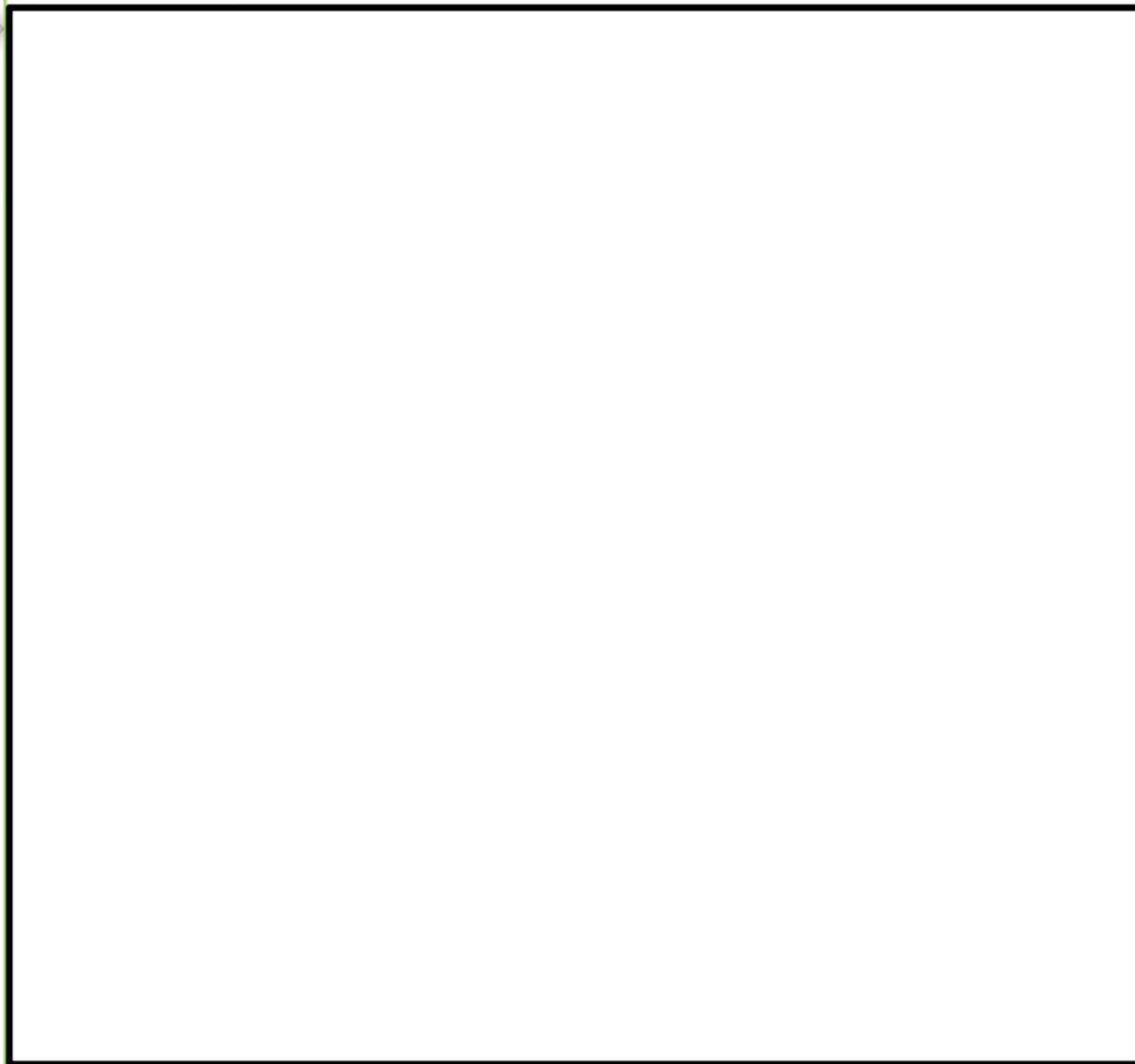**a.** Let $m =$ the number of pounds of mixed nuts. Complete the following table.

	Pounds	Price per Pound	Total Price
Dried Fruit	10	$6.25	6.25(10)
Mixed Nuts	☐	☐	☐
Trail Mix	☐	☐	☐

b. Write a rational equation using the last column of the table. ☐

c. Solve the equation to determine how many pounds of mixed nuts are needed. ☐

ROWING Sandra is rowing a canoe on Stanhope Lake. Her rate in still water is 6 miles per hour. It takes Sandra 3 hours to travel 10 miles round trip. Assuming that Sandra rowed at a constant rate of speed, determine the rate of the current.

Example 4

10. DISTANCE Alicia's average speed riding her bike is 11.5 miles per hour. She takes a round trip of 40 miles. It takes her 1 hour and 20 minutes with the wind and 2 hours and 30 minutes against the wind.

a. Write an expression for Alicia's time with the wind.

b. Write an expression for Alicia's time against the wind.

c. How long does it take to complete the trip?

d. Write and solve the rational equation to determine the speed of the wind.

COMMUNITY SERVICE Every year, the junior and senior classes at Hillcrest High School build a house for the community. If it takes the senior class 24 days to complete a house and 18 days if they work with the junior class, how long would it take the junior class to complete a house if they worked alone?

Understand We are given how long it takes the senior class working alone and when the classes work together. We need to determine how long it would take the junior class by themselves.

Plan The senior class can complete 1 house in 24 days, so their rate is $\frac{1}{24}$ of a house per day.

The rate for the junior class is $\frac{1}{j}$.

The combined rate for both classes is $\frac{1}{18}$.

Senior Rate	Junior Rate	Combined Rate
$\frac{1}{24}$	$\frac{1}{j}$	$\frac{1}{18}$

Solve

$$\frac{1}{24} + \frac{1}{j} = \frac{1}{18}$$ Write the equation.

$$72j\frac{1}{24} + 72j\frac{1}{j} = 72j\frac{1}{18}$$ LCD = 72j
Multiply by LCD.

$$\overset{3}{72}j\frac{1}{\underset{1}{24}} + 72\underset{1}{j}\frac{1}{\underset{1}{j}} = \overset{4}{72}j\frac{1}{\underset{1}{18}}$$ Divide common factors.

$$3j + 72 = 4j$$ Distribute.

Handwritten annotations:
1 day...

$$\frac{1}{24} + \frac{1}{j} = \frac{1}{18}$$

Example 5

11. WORK Kendal and Chandi wax cars. Kendal can wax a particular car in 60 minutes and Chandi can wax the same car in 80 minutes. They plan on waxing the same car together and want to know how long it will take.

a. How much will Kendal complete in 1 minute? $\dfrac{1}{60}$

b. How much will Kendal complete in x minutes? $\dfrac{x}{60}$

c. How much will Chandi complete in 1 minute? $\dfrac{1}{80}$

d. How much will Chandi complete in x minutes? $\dfrac{x}{80}$

e. Write a rational equation representing Kendal and Chandi working together on the car. $\dfrac{x}{60} + \dfrac{x}{80} = 1$

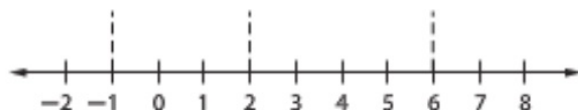**f.** Solve the equation to determine how long it will take them to finish the car. **about 34.3 min**

$$40x + 30x$$
$$70x = 240$$
$$4x + 3x = 240$$
$$7x = 240$$
$$x = \frac{240}{70}$$
$$x = 34.3$$

Example 6 Solve a Rational Inequality

Solve $\frac{x}{3} - \frac{1}{x-2} < \frac{x+1}{4}$.

Step 1 The excluded value for this inequality is 2.

Step 2 Solve the related equation.

$$\frac{x}{3} - \frac{1}{x-2} = \frac{x+1}{4} \qquad \text{Related equation}$$

$$\overset{4}{\cancel{12}}(x-2)\frac{x}{\cancel{3}}_1 - 12(x-2)\frac{1}{x-2} = \overset{3}{\cancel{12}}(x-2)\frac{x+1}{\cancel{4}}_1 \qquad \begin{array}{l}\text{LCD is } 12(x-2).\\ \text{Multiply by LCD.}\end{array}$$

$$4x^2 - 8x - 12 = 3x^2 - 3x - 6 \qquad \text{Distribute.}$$

$$x^2 - 5x - 6 = 0 \qquad \text{Subtract } 3x^2 - 3x - 6.$$

$$(x-6)(x+1) = 0 \qquad \text{Factor.}$$

$$x = 6 \text{ or } -1 \qquad \text{Zero Product Property}$$

Step 3 Draw vertical lines at the excluded value and at the solutions to separate the number line into intervals.

Step 4 Now test a sample value in each interval to determine whether the values in the interval satisfy the inequality.

Test $x = -3$.
$$\frac{-3}{3} - \frac{1}{-3-2} \overset{?}{<} \frac{-3+1}{4}$$
$$-1 + \frac{1}{5} \overset{?}{<} -\frac{2}{4}$$
$$-\frac{4}{5} < -\frac{1}{2} \checkmark$$

Test $x = 0$.
$$\frac{0}{3} - \frac{1}{0-2} \overset{?}{<} \frac{0+1}{4}$$
$$0 + \frac{1}{2} \overset{?}{<} \frac{1}{4}$$
$$\frac{1}{2} \not< \frac{1}{4}$$

Test $x = 4$.
$$\frac{4}{3} - \frac{1}{4-2} \overset{?}{<} \frac{4+1}{4}$$
$$\frac{4}{3} - \frac{1}{2} \overset{?}{<} \frac{5}{4}$$
$$\frac{5}{6} < \frac{5}{4} \checkmark$$

Test $x = 8$.
$$\frac{8}{3} - \frac{1}{8-2} \overset{?}{<} \frac{8+1}{4}$$
$$\frac{32}{12} - \frac{2}{12} \overset{?}{<} \frac{27}{12}$$
$$\frac{30}{12} \not< \frac{27}{12}$$

The statement is true for $x = -3$ and $x = 4$. Therefore, the solution is $x < -1$ or $2 < x < 6$.

Example 6 Solve each inequality. Check your solutions.

12. $\dfrac{3}{5x} + \dfrac{1}{6x} > \dfrac{2}{3}$ [ ]

13. $\dfrac{1}{4c} + \dfrac{1}{9c} < \dfrac{1}{2}$ $c < 0$, or $\dfrac{13}{18} < c$

14. $\dfrac{4}{3y} + \dfrac{2}{5y} < \dfrac{3}{2}$ [ ]

15. $\dfrac{1}{3b} + \dfrac{1}{4b} < \dfrac{1}{5}$ $b < 0$, or $\dfrac{35}{12} < b$

Examples 1–2 Solve each equation. Check your solutions.

16. $\dfrac{9}{x-7} - \dfrac{7}{x-6} = \dfrac{13}{x^2-13x+42}$ **9**

17. $\dfrac{13}{y+3} - \dfrac{12}{y+4} = \dfrac{18}{y^2+7y+12}$ **2**

18. $\dfrac{14}{x-2} - \dfrac{18}{x+1} = \dfrac{22}{x^2-x-2}$ **7**

19. $\dfrac{11}{a+2} - \dfrac{10}{a+5} = \dfrac{36}{a^2+7a+10}$ **1**

20. $\dfrac{x}{2x-1} + \dfrac{3}{x+4} = \dfrac{21}{2x^2+7x-4}$ **−12, 2**

21. $\dfrac{2}{y-5} + \dfrac{y-1}{2y+1} = \dfrac{2}{2y^2-9y-5}$ **Ø**

Examples 3–5 **22. CHEMISTRY** How many milliliters of a 20% acid solution must be added to 40 milliliters of a 75% acid solution to create a 30% acid solution? **180 mL**

23 **GROCERIES** Ellen bought 3 pounds of bananas for $0.90 per pound. How many pounds of apples costing $1.25 per pound must she purchase so that the total cost for fruit is $1 per pound? **1.2 lb**

24. BUILDING Bryan's volunteer group can build a garage in 12 hours. Sequoia's group can build it in 16 hours. How long would it take them if they worked together? **about 6.86 hours**

Example 6 Solve each inequality. Check your solutions.

25. $3 - \dfrac{4}{x} > \dfrac{5}{4x}$ **x < 0 or x > 1.75**

26. $\dfrac{5}{3a} - \dfrac{3}{4a} > \dfrac{5}{6}$ **0 < a < 1.1**

27. $\dfrac{x-2}{x+2} + \dfrac{1}{x-2} > \dfrac{x-4}{x-2}$ **x < −2, or 2 < x < 14**

28. $\dfrac{3}{4} - \dfrac{1}{x-3} > \dfrac{x}{x+4}$ **−4 < x < 3**

29. $\dfrac{x}{5} + \dfrac{2}{3} < \dfrac{3}{x-4}$ **x < −5 or 4 < x < $\dfrac{17}{3}$**

30. $\dfrac{x}{x+2} + \dfrac{1}{x-1} < \dfrac{3}{2}$ **2 < x, −2 < x < 1, x < −5**