3-8 Solving Systems of Equations Using Inverse Matrices

1 Identity and Inverse Matrices

Recall that in real numbers, two numbers are multiplicative inverses if their product is the multiplicative identity, 1. Similarly, for matrices, the **identity matrix** is a *square matrix* that, when multiplied by another matrix, equals that same matrix. A **square matrix** is a matrix with the same number of rows and columns.

2 × 2 Identity Matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3 × 3 Identity Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

KeyConcept Identity Matrix for Multiplication

Words The identity matrix for multiplication I is a square matrix with 1 for every element of the main diagonal, from upper left to lower right, and 0 in all other positions. For any square matrix A of the same dimension as I, $A \cdot I = I \cdot A = A$.

Symbols If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Two $n \times n$ matrices are **inverses** of each other if their product is the identity matrix. If matrix A has an inverse symbolized by A^{-1}, then $A \cdot A^{-1} = A^{-1} \cdot A = I$.

Determine whether the matrices in each pair are inverses.

a. $A = \begin{bmatrix} -4 & 2 \\ -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ \frac{1}{2} & -1 \end{bmatrix}$

If A and B are inverses, then $A \cdot B = B \cdot A = I$.

$$A \cdot B = \begin{bmatrix} -4 & 2 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ \frac{1}{2} & -1 \end{bmatrix}$$ Write an equation.

$$= \begin{bmatrix} -1+1 & 2-2 \\ -\frac{1}{2}+\frac{1}{2} & 1-1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ Matrix multiplication

Since $A \cdot B \neq I$, they are *not* inverses.

b. $F = \begin{bmatrix} 3 & -5 \\ -2 & 6 \end{bmatrix}$ and $G = \begin{bmatrix} \frac{3}{4} & \frac{5}{8} \\ \frac{1}{4} & \frac{3}{8} \end{bmatrix}$

If F and G are inverses, then $F \cdot G = G \cdot F = I$.

$$F \cdot G = \begin{bmatrix} 3 & -5 \\ -2 & 6 \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{4} & \frac{5}{8} \\ \frac{1}{4} & \frac{3}{8} \end{bmatrix}$$ Write an equation.

$$= \begin{bmatrix} \frac{9}{4}-\frac{5}{4} & \frac{15}{8}-\frac{15}{8} \\ -\frac{6}{4}+\frac{6}{4} & -\frac{10}{8}+\frac{18}{8} \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ Matrix multiplication

$$G \cdot F = \begin{bmatrix} \frac{3}{4} & \frac{5}{8} \\ \frac{1}{4} & \frac{3}{8} \end{bmatrix} \cdot \begin{bmatrix} 3 & -5 \\ -2 & 6 \end{bmatrix}$$ Write an equation.

$$= \begin{bmatrix} \frac{9}{4}-\frac{10}{8} & -\frac{15}{4}+\frac{30}{8} \\ \frac{3}{4}-\frac{6}{8} & -\frac{5}{4}+\frac{18}{8} \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ Matrix multiplication

Since $F \cdot G = G \cdot F = I$, F and G are inverses.

Example 1 **Determine whether the matrices in each pair are inverses.**

1. $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ no

2. $C = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}, D = \begin{bmatrix} 2 & 1 \\ 5 & -3 \end{bmatrix}$ no

3. $F = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, G = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$ yes

4. $H = \begin{bmatrix} -3 & -1 \\ -4 & -2 \end{bmatrix}, J = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$ no

3 $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} (-1)(-1) + (1)(0) & (-1)(-1) + (1)(-1) \\ 1 & 0 \\ (0)(-1) + (-1)(0) & (0)(-1) + (-1)(-1) \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 \end{bmatrix}$

2) $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & -3 \end{bmatrix}$

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$= \begin{bmatrix} (2)(2)+(1)(5) & (2)(1)+(1)(-3) \\ (5)(2)+(3)(5) & (5)(1)+(5)(-3) \end{bmatrix}$

$= \begin{bmatrix} 4+5 & 2+-3 \\ 10+15 & 5+-15 \end{bmatrix} = \begin{bmatrix} 9 & -1 \\ 25 & -10 \end{bmatrix} =$

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KeyConcept Inverse of a 2 × 2 Matrix

The inverse of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$.

Find the inverse of each matrix, if it exists.

a. $P = \begin{bmatrix} 7 & -5 \\ 2 & -1 \end{bmatrix}$

$\begin{vmatrix} 7 & -5 \\ 2 & -1 \end{vmatrix} = -7 - (-10)$ or 3 — Find the determinant.

Since the determinant does not equal 0, P^{-1} exists.

$P^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ — Definition of inverse

$= \frac{1}{7(-1) - (-5)(2)}\begin{bmatrix} -1 & 5 \\ -2 & 7 \end{bmatrix}$ — $a = 7, b = -5, c = 2, d = -1$

KeyConcept Inverse of a 2 × 2 Matrix

The inverse of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$.

Example 2 **Find the inverse of each matrix, if it exists.** **5–8. See margin.**

5. $\begin{bmatrix} 6 & -3 \\ -1 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 2 & -4 \\ -3 & 0 \end{bmatrix}$

7. $\begin{bmatrix} -3 & 0 \\ 5 & 2 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$

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$$\frac{1}{(6)(0)-(-3)(-1)}\begin{bmatrix} 0 & 3 \\ 1 & 6 \end{bmatrix}$$

$$\frac{1}{-3}\begin{bmatrix} 0 & 3 \\ 1 & 6 \end{bmatrix}$$

5. $\begin{bmatrix} 0 & -1 \\ -\frac{1}{3} & -2 \end{bmatrix}$

6. $\begin{bmatrix} 0 & -\frac{1}{3} \\ -\frac{1}{4} & -\frac{1}{6} \end{bmatrix}$

7. $\begin{bmatrix} -\frac{1}{3} & 0 \\ \frac{5}{6} & \frac{1}{2} \end{bmatrix}$

8. does not exist

2 Matrix Equations

Matrices can be used to represent and solve systems of equations. You can write a **matrix equation** to solve the system of equations below.

$$\begin{aligned} x + 2y &= 9 \\ 3x - 6y &= 3 \end{aligned} \quad \rightarrow \quad \begin{bmatrix} x + 2y \\ 3x - 6y \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$$

Write the left side of the matrix equation as the product of the coefficient matrix and the variable matrix. Write the right side as a constant matrix.

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & -6 \end{bmatrix}} \cdot \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{B}{\begin{bmatrix} 9 \\ 3 \end{bmatrix}}$$

Then solve the matrix equation in the same way that you would solve any other equation.

$ax = b$	Write the equation.	$AX = B$
$\left(\frac{1}{a}\right)ax = \left(\frac{1}{a}\right)b$	Multiply each side by the inverse of the coefficient, if it exists.	$A^{-1}AX = A^{-1}B$
$1x = \frac{b}{a}$	$\left(\frac{1}{a}\right)a = 1$, $A^{-1}A = I$	$IX = A^{-1}B$
$x = \frac{b}{a}$	$1x = x$, $IX = X$	$X = A^{-1}B$

KeyConcept Inverse of a 2 × 2 Matrix

The inverse of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$.

The inverse of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$.

Example 3 **Use a matrix equation to solve each system of equations.**

9. $-2x + y = 9$ **(−2, 5)**
$x + y = 3$

10. $4x - 2y = 22$ **(4, −3)**
$6x + 9y = -3$

11. $-2x + y = -4$ **(1, −2)**
$3x + y = 1$

12. **MONEY** Kevin had 25 quarters and dimes. The total value of all the coins was \$4. How many quarters and dimes did Kevin have? **10 quarters and 15 dimes**

$$-\frac{1}{3}\begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 9 \\ 3 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{(-2)(1)-(1)(1)}\begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix}}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 9 \\ 3 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} 9\cdot 1 + (-1)\cdot 3 \\ -1\cdot 9 + (-2)\cdot 3 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} 6 \\ -15 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

Example 1 **Determine whether each pair of matrices are inverses of each other.**

13. $K = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, L = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ **no**

14. $M = \begin{bmatrix} 0 & 2 \\ 4 & 5 \end{bmatrix}, N = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ **no**

15. $P = \begin{bmatrix} 4 & 0 \\ 3 & 0 \end{bmatrix}, Q = \begin{bmatrix} -1 & -1 \\ \frac{2}{3} & 5 \end{bmatrix}$ **no**

16. $R = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{2} \end{bmatrix}, S = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ **no**

Example 2 **Find the inverse of each matrix, if it exists.** **17–25. See margin.**

17. $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

18. $\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$

19. $\begin{bmatrix} 3 & 0 \\ 5 & 1 \end{bmatrix}$

20. $\begin{bmatrix} 1 & -1 \\ -6 & -1 \end{bmatrix}$

21 $\begin{bmatrix} -5 & -4 \\ 4 & 2 \end{bmatrix}$

22. $\begin{bmatrix} -5 & 9 \\ 4 & -8 \end{bmatrix}$

23. $\begin{bmatrix} 6 & -5 \\ 4 & 9 \end{bmatrix}$

24. $\begin{bmatrix} -4 & -2 \\ 7 & 8 \end{bmatrix}$

25. $\begin{bmatrix} -6 & 8 \\ 8 & -7 \end{bmatrix}$

Example 3

26. 6 mL of the red food coloring and 19 mL of the blue food coloring

26. **BAKING** Peggy is preparing a colored frosting for a cake. For the right shade of purple, she needs 25 milliliters of a 44% concentration food coloring. The store has a 25% red and a 50% blue concentration of food coloring. How many milliliters each of blue food coloring and red food coloring should be mixed to make the necessary amount of purple food coloring?

CCSS PERSEVERANCE **Use a matrix equation to solve each system of equations.**

27. $-x + y = 4$, $-x + y = -4$ **no solution**

28. $-x + y = 3$, $-2x + y = 6$ **$(-3, 0)$**

29. $x + y = 4$, $-4x + y = 9$ **$(-1, 5)$**

30. $3x + y = 3$, $5x + 3y = 6$ **$\left(\frac{3}{4}, \frac{3}{4}\right)$**

31. $y - x = 5$, $2y - 2x = 8$ **no solution**

32. $4x + 2y = 6$, $6x - 3y = 9$ **$(1.5, 0)$**

33. $1.6y - 0.2x = 1$, $0.4y - 0.1x = 0.5$ **$(-5, 0)$**

34. $4y - x = -2$, $3y - x = 6$ **$(-30, -8)$**

35. $2y - 4x = 3$, $4x - 3y = -6$ **$\left(\frac{3}{4}, 3\right)$**