3.9 Derivatives of Exponential and Logarithmic Functions

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Suppose we derive e^x....

$$\frac{d}{dx}(e^x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h\to 0} \frac{e^x \cdot e^h - e^x}{h}$$

$$= \lim_{h\to 0} \left(e^x \cdot \frac{e^h - 1}{h} \right)$$

$$= e^x \cdot \lim_{h\to 0} \left(\frac{e^h - 1}{h} \right)$$

$$= e^x \cdot 1$$

$$= e^x$$

This was used;

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

In other words, the derivative of this particular function is itself!

$$\frac{d}{dx}(e^x) = e^x$$

EXAMPLE 1 Using the Formula

otice

Rule is

Find dy/dx if $y = e^{(x+x^2)}$.

SOLUTION

Let $u = x + x^2$ then $y = e^u$. Then

$$\frac{dy}{dx} = e^u \frac{du}{dx}, \qquad \text{and} \qquad \frac{du}{dx} = 1 + 2x.$$

Thus, $\dfrac{dy}{dx} = e^u \dfrac{du}{dx} = e^{(x+x^2)}(1 + 2x)$.

Now try Exercise 9.

In Exercises 1–28, find dy/dx.

1. $y = 2e^x$ $\quad 2e^x$
2. $y = e^{2x}$ $\quad 2e^{2x}$
3. $y = e^{-x}$ $\quad -e^{-x}$
4. $y = e^{-5x}$ $\quad -5e^{-5x}$
5. $y = e^{2x/3}$ $\quad \frac{2}{3}e^{2x/3}$
6. $y = e^{-x/4}$ $\quad -\frac{1}{4}e^{-x/4}$
7. $y = xe^2 - e^x$ $\quad e^2 - e^x$
8. $y = x^2e^x - xe^x$ $\quad x^2e^x + xe^x - e^x$
9. $y = e^{\sqrt{x}}$ $\quad e^{\sqrt{x}}/2\sqrt{x}$
10. $y = e^{(x^2)}$

$e^{x^{\frac{1}{2}}}$

constant!

7 $\quad x e^2 - e^x$

$7.38x - e^x$

$\frac{d}{dx} e^u = e^u \cdot u'$

$u = 2x$

$u' = 2$

$e^{2x} \cdot 2$

9. $y = e^{\sqrt{x}}$ $\quad e^{\sqrt{x}}/2\sqrt{x}$

$e^{x^{\frac{1}{2}}}$

$$\frac{d}{dx} e^{x^{1/2}} = \overbrace{\left(e^{\sqrt{x}}\right)}^{e^u}\overbrace{\left(\frac{1}{2\sqrt{x}}\right)}^{u'} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

$u = x^{\frac{1}{2}}$

$$u' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$$

What if we are deriving an exponential with *any* base...?

$4^x = e^{x \ln 4} = e^{1.3x}$

constant!

First, we can rewrite exponentials using logarithmic properties;

$$a^x = e^{x \ln a}.$$

$e^{x \ln a} = e^{\ln(a^x)} = a^x$

$u = x \ln a$

$u' = \ln a$

We can then find the derivative of a^x with the Chain Rule.

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x \ln a} = e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) = e^{x \ln a} \cdot \ln a = a^x \ln a$$

Thus, if u is a differentiable function of x, we get the following rule.

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx}.$$

8. $y = x^2e^x - xe^x$ $x^2e^x + xe^x - e^x$

$f = x^2 \quad g = e^x$

$f' = 2x \quad g' = e^x$

$f = x \quad g = e^x$

$f' = 1 \quad g' = e^x$

$y' = 2x \cdot e^x + x^2 \cdot e^x - (1 \cdot e^x + 1x \cdot e^x)$

$x^2e^x + xe^x - e^x$

EXAMPLE 2 Reviewing the Algebra of Logarithms

At what point on the graph of the function $y = 2^t - 3$ does the tangent line have slope 2

SOLUTION

The slope is the derivative:

$$\frac{d}{dt}(2^t - 3) = 2^t \cdot \ln 2 - 0 = 2^t \ln 2.$$

We want the value of t for which … on the calcula-
tor, but we will use logarithms fo…

$$\frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx}$$

$$2^t \ln 2 = 21$$

$$2^t = \frac{21}{\ln 2}$$

$$\ln 2^t = \ln\left(\frac{21}{\ln 2}\right) \quad \text{Logarithm of both sides}$$

$$t \cdot \ln 2 = \ln 21 - \ln(\ln 2) \quad \text{Properties of logarithms}$$

$$t = \frac{\ln 21 - \ln(\ln 2)}{\ln 2}$$

$$t \approx 4.921$$

$$y = 2^t - 3 \approx 27.297 \quad \text{Using the stored value of } t$$

$\frac{d}{dx}\csc x = -\csc x \cot x$ $\frac{d}{dx}[\cot x] = -\csc^2 x$

In Exercises 1–28, find dy/dx.

$9^{-x}(\ln 9)(-1)$

11. $y = 8^x$ $8^x \ln 8$

12. $y = 9^{-x}$ $-9^{-x} \ln 9$

13. $y = 3^{\csc x}$

14. $y = 3^{\cot x}$ $-3^{\cot x}(\ln 3)(\csc^2 x)$

13 $y' = -3^{\csc x} \ln 3 (\csc x \cot x)$

$y = 3^{\csc x}$

a

$u = \csc x$

$y' = (3^{\csc x}) \cdot \ln 3 (-\csc x \cot x)$

11 $y = 8^x$

$y' = 8^x \ln 8 \cdot (1)$

$$\frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx}$$