3.9 Derivatives of Exponential and Logarithmic Functions

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Suppose we derive e^x....

$$\begin{aligned}
\frac{d}{dx}(e^x) &= \lim_{h\to 0} \frac{e^{x+h}-e^x}{h} \\
&= \lim_{h\to 0} \frac{e^x \cdot e^h - e^x}{h} \\
&= \lim_{h\to 0} \left(e^x \cdot \frac{e^h-1}{h}\right) \\
&= e^x \cdot \lim_{h\to 0} \left(\frac{e^h-1}{h}\right) \\
&= e^x \cdot 1 \\
&= e^x
\end{aligned}$$

This was used;

$$\lim_{h\to 0} \frac{e^h-1}{h} = 1.$$

In other words, the derivative of this particular function is itself!

$$\frac{d}{dx}(e^x) = e^x$$

EXAMPLE 1 Using the Formula

Find dy/dx if $y = e^{(x+x^2)}$.

SOLUTION

Let $u = x + x^2$ then $y = e^u$. Then

$$\frac{dy}{dx} = e^u \frac{du}{dx}, \qquad \text{and} \qquad \frac{du}{dx} = 1 + 2x.$$

Thus, $\dfrac{dy}{dx} = e^u \dfrac{du}{dx} = e^{(x+x^2)}(1 + 2x)$.

Now try Exercise 9.

otice

Rule is

In Exercises 1–28, find dy/dx.

1. $y = 2e^x$ $\quad 2e^x$
2. $y = e^{2x}$ $\quad 2e^{2x}$
3. $y = e^{-x}$ $\quad -e^{-x}$
4. $y = e^{-5x}$ $\quad -5e^{-5x}$
5. $y = e^{2x/3}$ $\quad \frac{2}{3}e^{2x/3}$
6. $y = e^{-x/4}$ $\quad -\frac{1}{4}e^{-x/4}$
7. $y = xe^2 - e^x$ $\quad e^2 - e^x$
8. $y = x^2e^x - xe^x$ $\quad x^2e^x + xe^x - e^x$
9. $y = e^{\sqrt{x}}$ $\quad e^{\sqrt{x}}/2\sqrt{x}$
10. $y = e^{(x^2)}$

$e^{x^{\frac{1}{2}}}$

constant!

7. $x e^2 - e^x$

$7.38x - e^x$

$\frac{d}{dx} e^u = e^u \cdot u'$

$u = 2x$

$u' = 2$

$e^{2x} \cdot 2$

9. $y = e^{\sqrt{x}}$ $\quad e^{\sqrt{x}}/2\sqrt{x}$

$e^{x^{\frac{1}{2}}}$

$$\frac{d}{dx} e^{x^{1/2}} = \underbrace{\left(e^{\sqrt{x}}\right)}_{e^u}\underbrace{\left(\frac{1}{2\sqrt{x}}\right)}_{u'} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

$u = x^{\frac{1}{2}}$

$u' = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$

What if we are deriving an exponential with *any* base...?

$4^x = e^{x \ln 4} = e^{1.3x}$

First, we can rewrite exponentials using logarithmic properties;

$$a^x = e^{x \ln a}.$$

constant!

$e^{x \ln a} = e^{\ln(a^x)} = a^x$

$u = x \ln a$

$u' = \ln a$

We can then find the derivative of a^x with the Chain Rule.

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x \ln a} = e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) = e^{x \ln a} \cdot \ln a = a^x \ln a$$

Thus, if u is a differentiable function of x, we get the following rule.

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx}.$$

8. $y = x^2e^x - xe^x$ $\quad x^2e^x + xe^x - e^x$

$f = x^2 \quad g = e^x$

$f' = 2x \quad g' = e^x$

$f = x \quad g = e^x$

$f' = 1 \quad g' = e^x$

$y' = 2x \cdot e^x + x^2 \cdot e^x - (1 \cdot e^x + 1 x \cdot e^x)$

$x^2e^x + xe^x - e^x$

EXAMPLE 2 Reviewing the Algebra of Logarithms

At what point on the graph of the function $y = 2^t - 3$ does the tangent line have slope 21

SOLUTION

The slope is the derivative:

$$\frac{d}{dt}(2^t - 3) = 2^t \cdot \ln 2 - 0 = 2^t \ln 2.$$

We want the value of t for which

$$\frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx}$$

on the calcula-
tor, but we will use logarithms fo

$$2^t \ln 2 = 21$$

$$2^t = \frac{21}{\ln 2}$$

$$\ln 2^t = \ln\left(\frac{21}{\ln 2}\right) \qquad \text{Logarithm of both sides}$$

$$t \cdot \ln 2 = \ln 21 - \ln(\ln 2) \qquad \text{Properties of logarithms}$$

$$t = \frac{\ln 21 - \ln(\ln 2)}{\ln 2}$$

$$t \approx 4.921$$

$$y = 2^t - 3 \approx 27.297 \qquad \text{Using the stored value of } t$$

$\frac{d}{dx}\csc x = -\csc x \cot x$ $\frac{d}{dx}[\cot x] = -\csc^2 x$

In Exercises 1–28, find dy/dx.

$9^{-x}(\ln 9)(-1)$

11. $y = 8^x$ $8^x \ln 8$

12. $y = 9^{-x}$ $-9^{-x}\ln 9$

13. $y = 3^{\csc x}$

14. $y = 3^{\cot x}$ $-3^{\cot x}(\ln 3)(\csc^2 x)$

13 $y' = -3^{\csc x}\ln 3(\csc x \cot x)$

11 $y = 8^x$

$y' = 8^x \ln 8 \cdot (1)$

$y = 3^{\csc x}$

a

$u = \csc x$

$y' = (3^{\csc x}) \cdot \ln 3(-\csc x \cot x)$

$$\frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx}.$$

Derivative of ln *x*

Now that we know the derivative of e^x, it is relatively easy to find the derivative of its inverse function, ln x.

$$y = \ln x = \log_e x$$

$$e^y = x \qquad \text{Inverse function relationship}$$

$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x) \qquad \text{Differentiate implicitly.}$$

$$e^y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

If u is a differentiable function of x and $u > 0$,

$$\boxed{\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}.}$$

In Exercises 1–28, find dy/dx.

$$\frac{u'}{u}$$

15. $y = \ln(x^2)$ $\frac{2}{x}$
16. $y = (\ln x)^2$ $\frac{2 \ln x}{x}$
17. $y = \ln(1/x)$ See page 180.
18. $y = \ln(10/x)$ See page 180.
19. $y = \ln(\ln x)$ $\frac{1}{x \ln x}$
20. $y = x \ln x - x$ $\ln x$

15

$u = x^2$

$u' = 2x$

$\frac{u'}{u} = \frac{2x}{x^2}$

16

$y = (\ln x)^2$

$y = u^2$ $u = \ln x$

$y' = 2u$ $u' = \frac{1}{x}$

$f'(u) \cdot u'$

$y' = 2u \cdot u'$

$= 2(\ln x)\left(\frac{1}{x}\right)$

17. $y = \ln(1/x)$ See page 180. **18.** $y = \ln(10/x)$ See page 180.

u

$u = \frac{10}{x} = 10x^{-1}$

$u' = -10x^{-2} = -\frac{10}{x^2}$

$\frac{u'}{u}$

$\frac{-\frac{10}{x^2}}{\frac{10}{x}} = -\frac{10}{x^2} \cdot \frac{x}{10}$

$-\frac{x}{x^2} = -\frac{1}{x}$

In Exercises 1–28, find dy/dx.

$\frac{d}{dx}\ln(u) = \frac{1}{u}\cdot u'$

15. $y = \ln(x^2)$ $\frac{2}{x}$

16. $y = (\ln x)^2$ $\frac{2\ln x}{x}$

17. $y = \ln(1/x)$ See page 180.

18. $y = \ln(10/x)$ See page 180.

19. $y = \ln(\ln x)$ $\frac{1}{x\ln x}$

20. $y = x\ln x - x$ $\ln x$

15. $\ln(x^2)$:

$u = x^2$

$u' = 2x$

$y = \ln(u)$

$y' = \frac{1}{u}$

16. $y = (\ln x)^2$

$y = u^2$

$y' = 2u$

$u = \ln x$

$u' = \frac{1}{x}$

$\frac{dy}{dx} = 2(\ln x)\cdot\frac{1}{x}$

Derivative of $\log_a x$

To find the derivative of $\log_a x$ for an arbitrary base ($a > 0$, $a \neq 1$), we use the change-of-base formula for logarithms to express $\log_a x$ in terms of natural logarithms, as follows:

$$\log_a x = \frac{\ln x}{\ln a}.$$

The rest is easy:

$$\begin{aligned}
\frac{d}{dx}\log_a x &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) \\
&= \frac{1}{\ln a} \cdot \frac{d}{dx}\ln x \qquad \text{Since } \ln a \text{ is a constant} \\
&= \frac{1}{\ln a} \cdot \frac{1}{x} \\
&= \frac{1}{x \ln a}.
\end{aligned}$$

So, if u is a differentiable function of x and $u > 0$, the formula is as follows.

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}\log_a u = \frac{1}{u \ln a}\frac{du}{dx}.$$

$$\frac{d}{dx}\log_a u = \frac{1}{u \ln a}\frac{du}{dx}.$$ $\leftarrow u'$

d dy/dx.

21. $y = \log_4 x^2$ See page 180.
22. $y = \log_5 \sqrt{x}$ See page 180.
23. $y = \log_2 (1/x)$ $-\frac{1}{x \ln 2}, x > 0$
24. $y = 1/\log_2 x$ $-\frac{1}{x(\ln 2)(\log_2 x)^2}$
25. $y = \ln 2 \cdot \log_2 x$ $\frac{1}{x}, x > 0$
26. $y = \log_3 (1 + x \ln 3)$

$f(x)$

$f(2)$

21 $u = x^2$

$u' = 2x$

$$\frac{d}{dx}\left[\log_4 u\right] = \frac{1}{u \ln 4} \cdot u'$$

$$= \frac{1 \cdot 2x}{x^2 \cdot \ln 4} = \frac{2\cancel{x}}{x^{\cancel{2}} \ln 4}$$

25. $y = \ln 2 \cdot \log_2 x$ $\frac{1}{x}, x > 0$

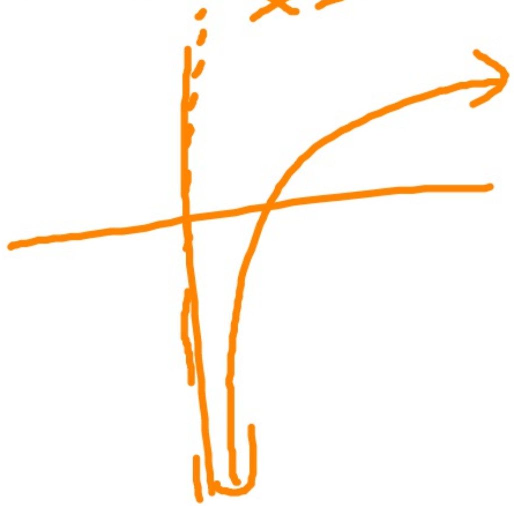
constant!

$$\frac{d}{dx}\log_a u = \frac{1}{u \ln a}\frac{du}{dx}.$$

$$\ln 2\left(\frac{d}{dx}\log_2 x\right)$$

$$\cancel{\ln 2}\left(\frac{1}{x\cancel{\ln 2}}\right)$$

$x > 0$



$$\frac{d}{dx}\log_a u = \frac{1}{u \ln a}\frac{du}{dx}.$$

22. $y = \log_5 \sqrt{x}$ See page 180.

$a = 5$

$u = x^{1/2}$

$u' = \frac{1}{2}x^{-1/2}$

$x^{1/2} \cdot x^{1/2}$

$x^{1/2 + 1/2}$

x^1

$$\frac{d}{dx}\left[\log_5 u\right]$$

$$= \frac{1}{(x^{1/2})(\ln 5)}\left(\frac{1}{2}x^{-1/2}\right)$$

$$= \frac{1}{(x^{1/2})(\ln 5)(2)(x^{1/2})}$$

$$= \frac{1}{2x \ln 5}$$

In Exercises 1–28, find dy/dx. $y = (\log_2 x)^{-1}$

21. $y = \log_4 x^2$ See page 180.

22. $y = \log_5 \sqrt{x}$ See page 180.

23. $y = \log_2 (1/x)$ $-\frac{1}{x \ln 2}, x > 0$

24. $y = 1/\log_2 x$ $-\frac{1}{x(\ln 2)(\log_2 x)^2}$

25. $y = \ln 2 \cdot \log_2 x$ $\frac{1}{x}, x > 0$

26. $y = \log_3 (1 + x \ln 3)$

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}\log_a u = \frac{1}{u \ln a}\frac{du}{dx}.$$

24 $(\log_2 x)^{-1}$

$u = \log_2 x$

$y = u^{-1}$

$u' = \frac{1}{x \ln 2} \cdot (1)$

$y' = -1u^{-2}$

$= \frac{-1}{u^2}$

$= \frac{-1}{(\log_2 x)^2} \cdot \frac{1}{x \ln 2}$

EXAMPLE 4 Going the Long Way with the Chain Rule

Find dy/dx if $y = \log_a a^{\sin x}$.

SOLUTION

Carefully working from the outside in, we apply the Chain Rule to get:

$$\frac{d}{dx}(\log_a a^{\sin x}) = \frac{1}{a^{\sin x}\ln a}\cdot\frac{d}{dx}(a^{\sin x}) \qquad \log_a u,\ u = a^{\sin x}$$

$$= \frac{1}{a^{\sin x}\ln a}\cdot a^{\sin x}\ln a\cdot\frac{d}{dx}(\sin x) \qquad a^u,\ u = \sin x$$

$$= \frac{a^{\sin x}\ln a}{a^{\sin x}\ln a}\cdot\cos x$$

$$= \cos x.$$

$\log_a a^{x} = x$

Now try Exercise 23.

OR, we could reduce first, then derive...

$y = \log_a a^{\sin x} = \sin x$

In Exercises 1–28, find dy/dx.

$\log a^x = x \cdot \log a$

27. $y = \log_{10} e^x$ $\frac{1}{\ln 10}$

28. $y = \ln 10^x$ $\ln 10$

y'

$y = (\ln 10) \cdot x$