4.1 Extreme Values of Functions

What you'll learn about

- Absolute (Global) Extreme Values
- Local (Relative) Extreme Values
- Finding Extreme Values

Absolute

biggest #

least #

"walls"

end points are used

local

end points are NOT used...

big #

small #

"general area"

DEFINITION Absolute Extreme Values

Let f be a function with domain D. Then $f(c)$ is the

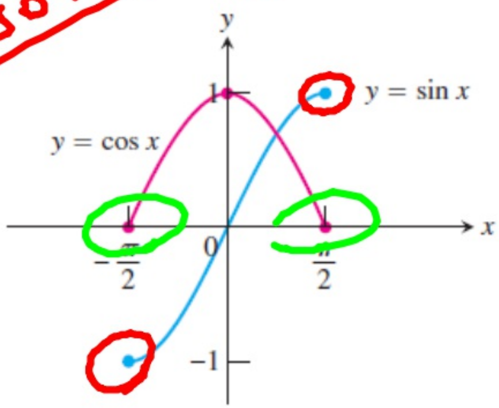
(a) **absolute maximum value** on D if and only if $f(x) \leq f(c)$ for all x in D.

(b) **absolute minimum value** on D if and only if $f(x) \geq f(c)$ for all x in D.

EXAMPLE 1 Exploring Extreme Values

On $[-\pi/2, \pi/2]$, $f(x) = \cos x$ takes on a maximum value of 1 (once) and a minimum value of 0 (twice). The function $g(x) = \sin x$ takes on a maximum value of 1 and a minimum value of -1 (Figure 4.1). *Now try Exercise 1.*

absolute extrema



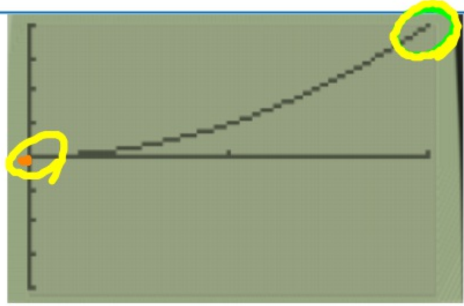
Please note that the endpoints were considered in the *closed* interval.

EXAMPLE 2 Exploring Absolute Extrema

The absolute extrema of the following functions on their domains can be seen in Figure 4.2.

	Function Rule	Domain D	Absolute Extrema on D
(a)	$y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at $x = 0$.
(b)	$y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$. Absolute minimum of 0 at $x = 0$.
(c)	$y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$. No absolute minimum.
(d)	$y = x^2$	$(0, 2)$	No absolute extrema.

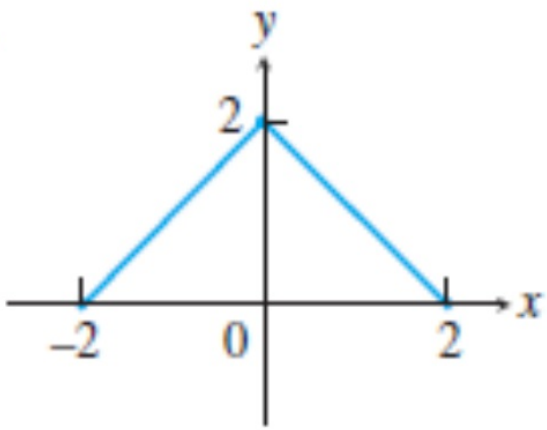
1.99... ?



Now try Exercise 3.

In Exercises 1–4, find the extreme values and where they occur.

1.



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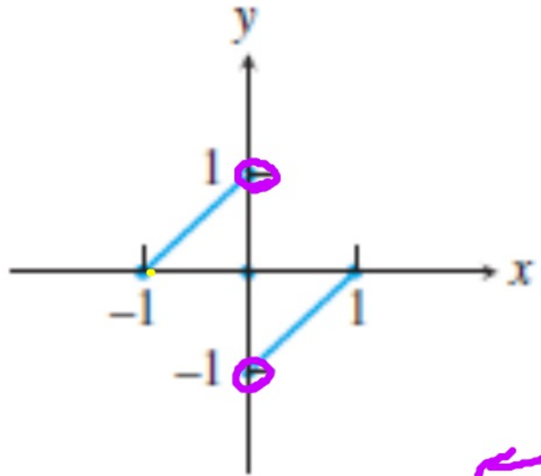
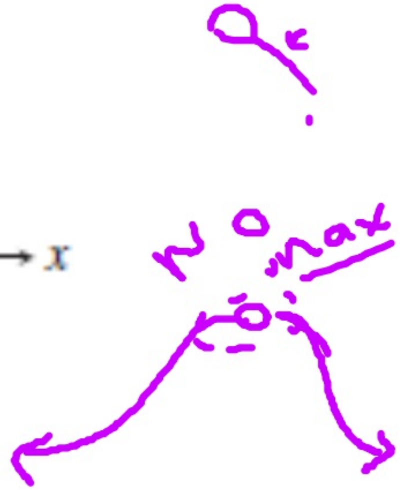
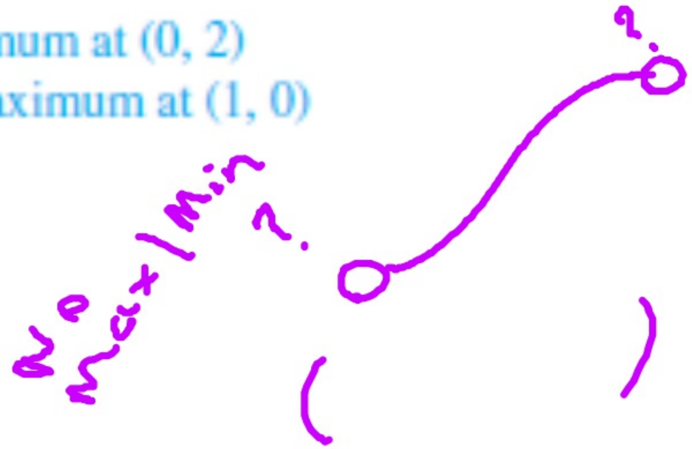


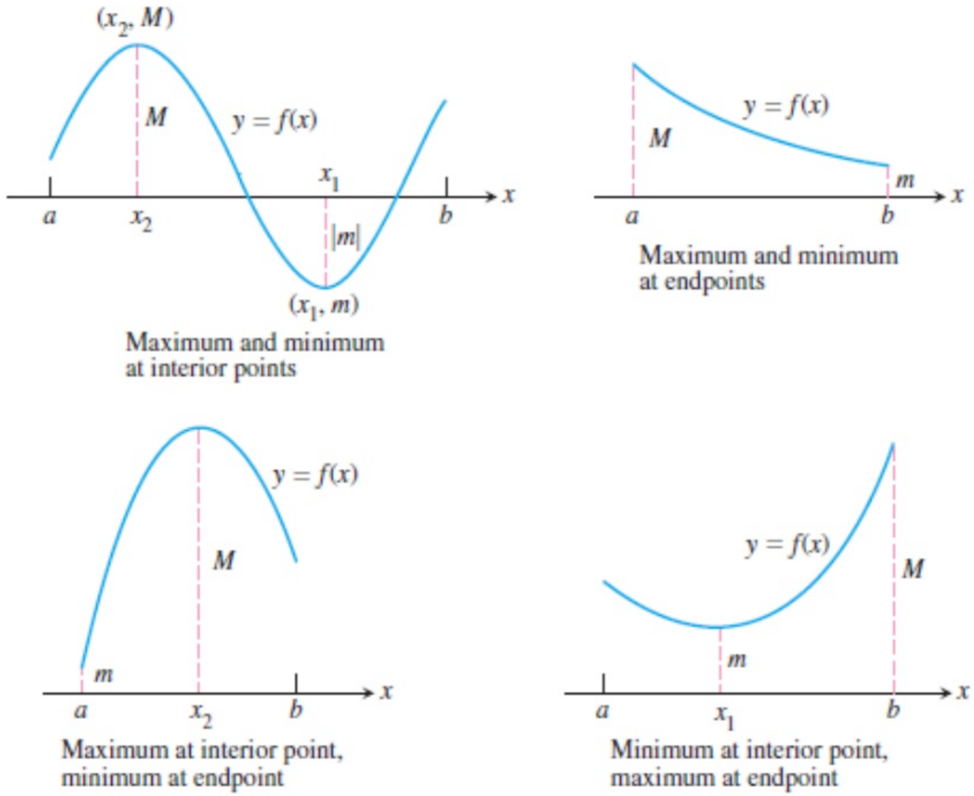


1. Minima at (–2, 0) and (2, 0), maximum at (0, 2)
2. Local minimum at (–1, 0), local maximum at (1, 0)



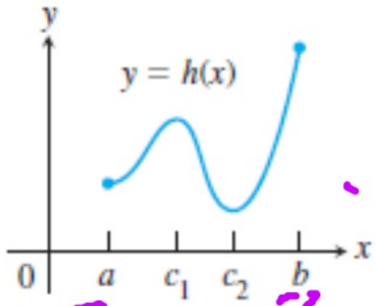
THEOREM 1 The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f has both a maximum value and a minimum value on the interval. (Figure 4.3)



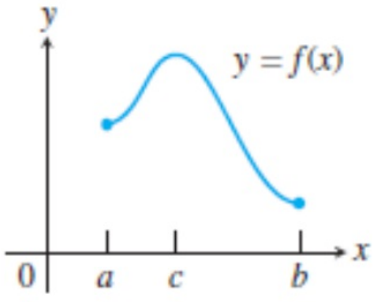
In Exercises 5–10, identify each x-value at which any absolute extreme value occurs. Explain how your answer is consistent with the Extreme Value Theorem. See page 195.

5.



[ ]
max

6.

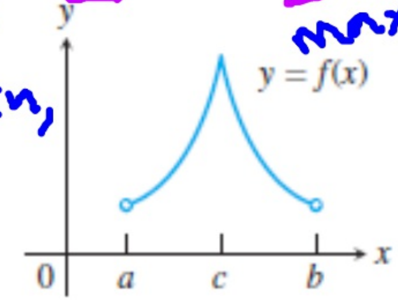


No end pts.

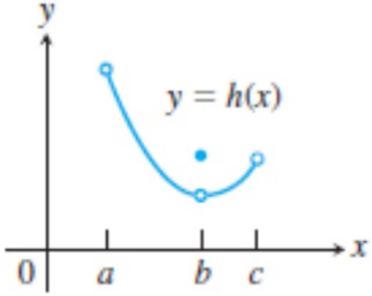
NOT contin-
uous

7.

No Min,

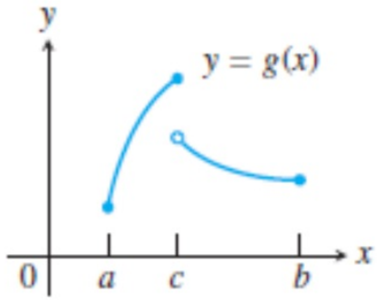


8.

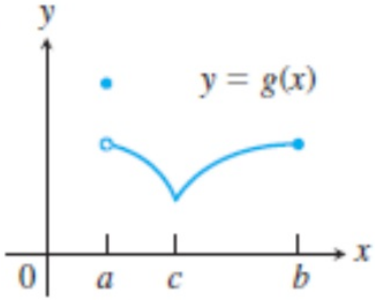


8 No Max,
No Min

9.



10.



DEFINITION **Local Extreme Values**

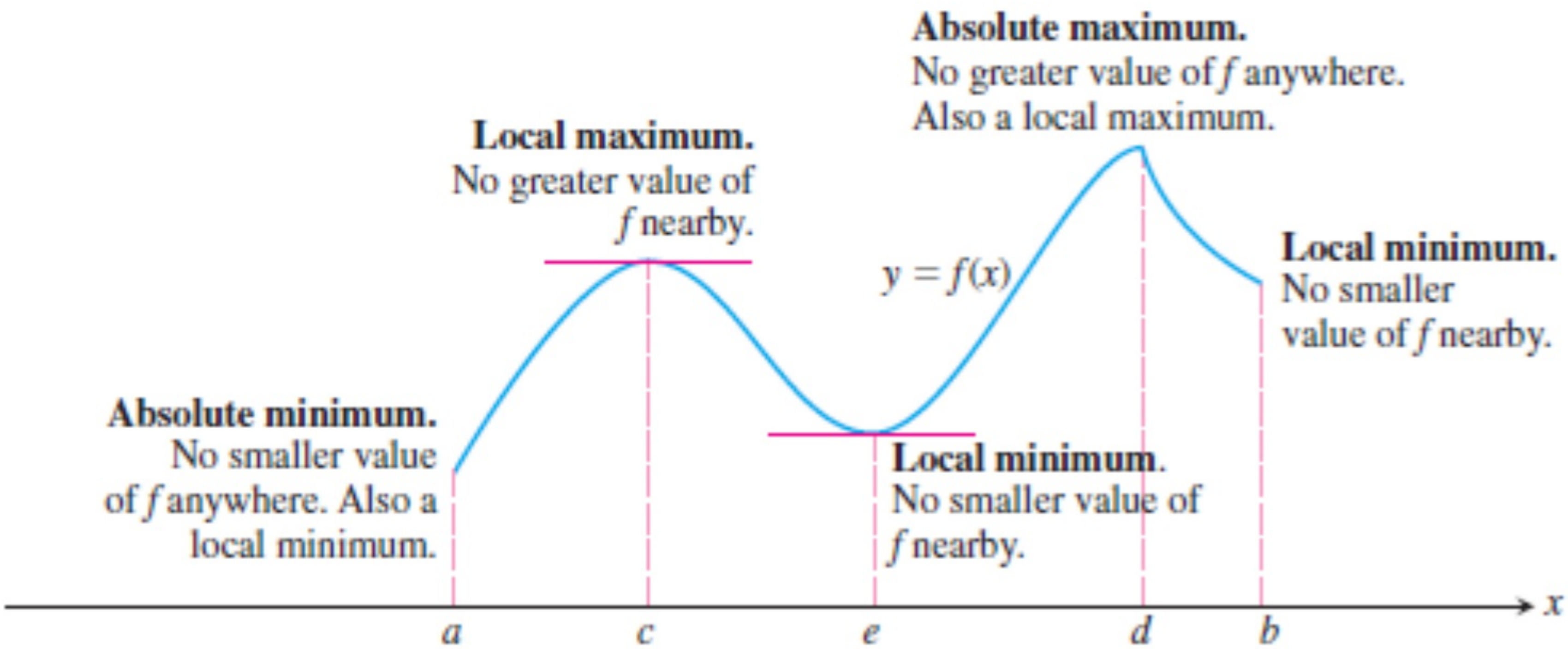
Let c be an interior point of the domain of the function f. Then $f(c)$ is a

(a) **local maximum value** at c if and only if $f(x) \leq f(c)$ for all x in some open interval containing c.

(b) **local minimum value** at c if and only if $f(x) \geq f(c)$ for all x in some open interval containing c.

A function f has a local maximum or local minimum *at an endpoint* c if the appropriate inequality holds for all x in some half-open domain interval containing c.

Note how relative extremas are considered around *open* intervals.



THEOREM 2 Local Extreme Values

If a function f has a local maximum value or a local minimum value at an interior point c of its domain, and if f' exists at c, then

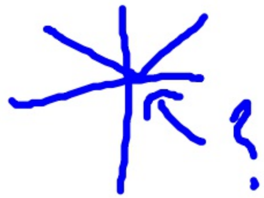
$$f'(c) = 0.$$

From this theorem, we are able to identify *critical points;*

DEFINITION Critical Point

A point in the interior of the domain of a function f at which $f' = 0$ or f' does not exist is a **critical point** of f.

Ex. recall the graph of f(x)=|x|...



EXAMPLE 3 Finding Absolute Extrema

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

closed → Test end points!

SOLUTION

Solve Graphically Figure 4.5 suggests that f has an absolute maximum value of about 2 at $x = 3$ and an absolute minimum value of 0 at $x = 0$.

Confirm Analytically We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

← can't be zero...

has no zeros but is undefined at $x = 0$. The values of f at this one critical point and at the endpoints are

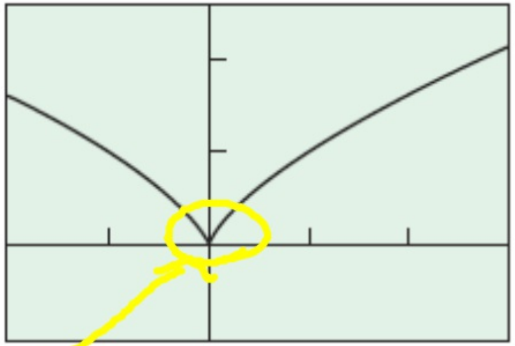
Critical point value: $f(0) = 0$;

Endpoint values: $f(-2) = (-2)^{2/3} = \sqrt[3]{4}$;

$f(3) = (3)^{2/3} = \sqrt[3]{9}$.

We can see from this list that the function's absolute maximum value is $\sqrt[3]{9} \approx 2.08$, and occurs at the right endpoint $x = 3$. The absolute minimum value is 0, and occurs at the interior point $x = 0$. *Now try Exercise 11.*

$y = x^{2/3}$



[–2, 3] by [–1, 2.5]

Figure 4.5 (Example 3)

In Exercises 11–18, use analytic methods to find the extreme values of the function on the interval and where they occur. See page 195.

11. $f(x) = \frac{1}{x} + \ln x, \quad 0.5 \le x \le 4$

12. $g(x) = e^{-x}, \quad -1 \le x \le 1$

13. $h(x) = \ln(x + 1), \quad 0 \le x \le 3$

14. $k(x) = e^{-x^2}, \quad -\infty < x < \infty$

15. $f(x) = \sin\left(x + \frac{\pi}{4}\right), \quad 0 \le x \le \frac{7\pi}{4}$

16. $g(x) = \sec x, \quad -\frac{\pi}{2} < x < \frac{3\pi}{2}$

17. $f(x) = x^{2/5}, \quad -3 \le x < 1$

18. $f(x) = x^{3/5}, \quad -2 < x \le 3$

13 $h'(x) = \frac{1}{x+1}$

$x \neq -1$

Abs. min

Test $x = 0$ and $x = 3$...

$\ln(1) = 0$

$\ln(4) = 1.4$

Abs. max.

$\ln(0) = $ ☹

18 $f'(x) = \frac{3}{5} x^{-2/5} = \frac{3}{5\sqrt[5]{x^2}}$

Min

$x \neq 0$

$x = 0$;

$f(0) = 0^{3/5} = 0$

$x = 3$;

$f(3) = 3^{3/5}$

≈ 3

≈ 1.9

Max

In Exercises 19–30, find the extreme values of the function and where they occur.

19. $y = 2x^2 - 8x + 9$ Min value 1 at $x = 2$

20. $y = x^3 - 2x + 4$

21. $y = x^3 + x^2 - 8x + 5$

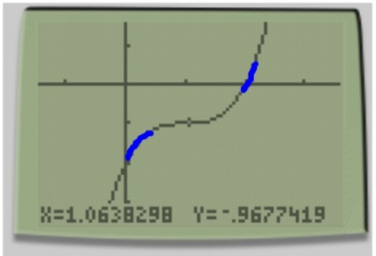
22. $y = x^3 - 3x^2 + 3x - 2$ None ?

$$y' = 3x^2 - 6x + 3 = (3x - 3)(x - 1)$$

$$3(x-1)(x-1)$$

$$f'(x) = 3(x-1)^2 = 0$$

No undefined points



$x = 1$

First Derivative Test

Test $x=0$ Test $x=2$

1

$f'(x) > 0$ $f'(x) > 0$

+ +

No sign change $\Rightarrow$ Not an extrema