4.1 Extreme Values of Functions

What you'll learn about

- Absolute (Global) Extreme Values
- Local (Relative) Extreme Values
- Finding Extreme Values

Absolute — biggest #, least # — "walls" — end points are used

local — (big #, small #) — "general area" — end points are NOT used...

DEFINITION Absolute Extreme Values

Let f be a function with domain D. Then $f(c)$ is the

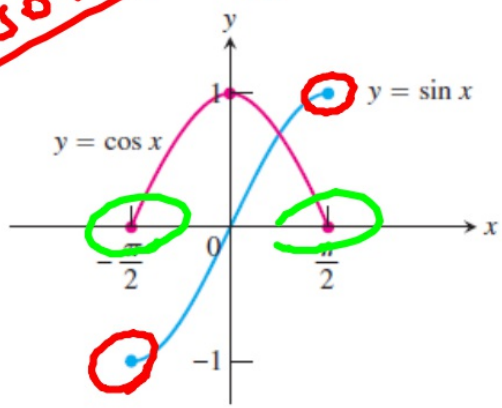
(a) **absolute maximum value** on D if and only if $f(x) \leq f(c)$ for all x in D.

(b) **absolute minimum value** on D if and only if $f(x) \geq f(c)$ for all x in D.

EXAMPLE 1 Exploring Extreme Values

On $[-\pi/2, \pi/2]$, $f(x) = \cos x$ takes on a maximum value of 1 (once) and a minimum value of 0 (twice). The function $g(x) = \sin x$ takes on a maximum value of 1 and a minimum value of -1 (Figure 4.1). ***Now try Exercise 1.***

absolute extrema

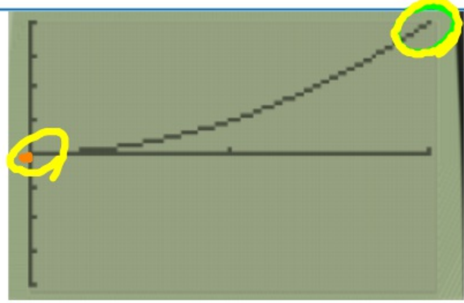


Please note that the endpoints were considered in the *closed* interval.

EXAMPLE 2 Exploring Absolute Extrema

The absolute extrema of the following functions on their domains can be seen in Figure 4.2.

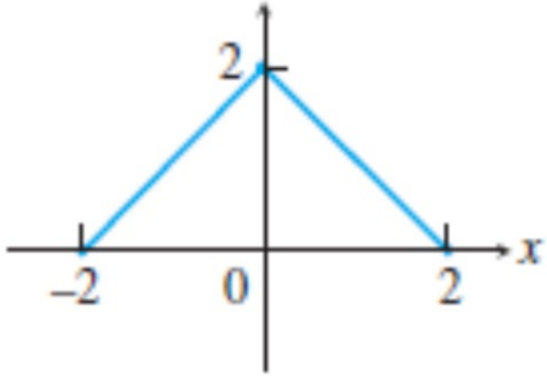
	Function Rule	Domain D	Absolute Extrema on D
(a)	$y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at $x = 0$.
(b)	$y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$. Absolute minimum of 0 at $x = 0$.
(c)	$y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$. No absolute minimum.
(d)	$y = x^2$	$(0, 2)$	No absolute extrema.



Now try Exercise 3.

In Exercises 1–4, find the extreme values and where they occur.

1.



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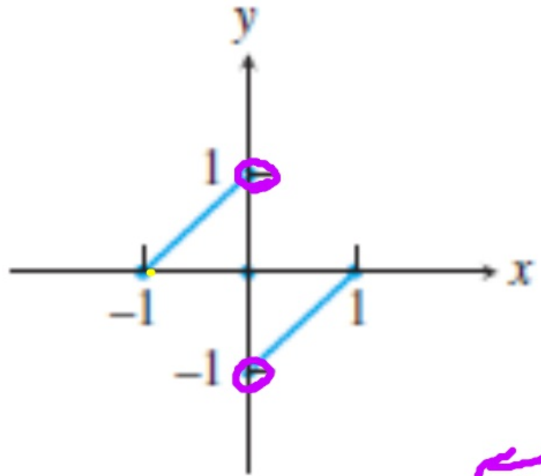
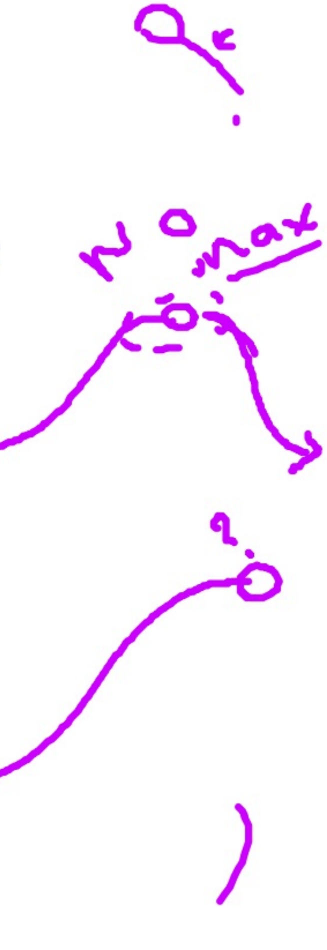
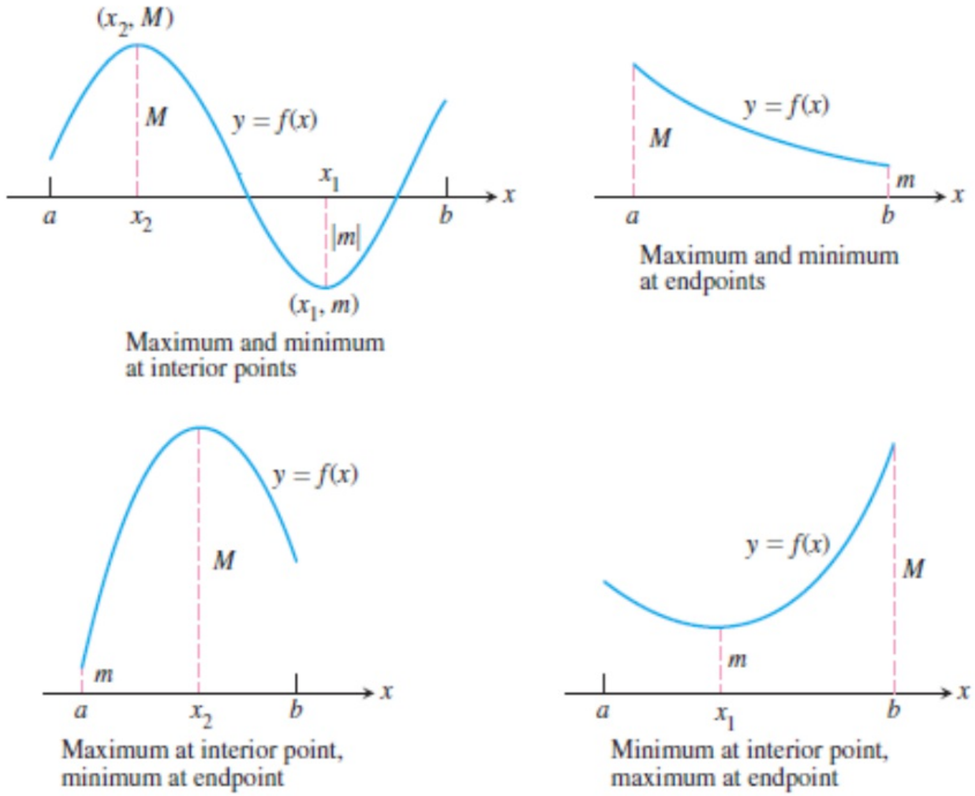




1. Minima at (−2, 0) and (2, 0), maximum at (0, 2)
2. Local minimum at (−1, 0), local maximum at (1, 0)

THEOREM 1 The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f has both a maximum value and a minimum value on the interval. (Figure 4.3)



Maximum and minimum at interior points

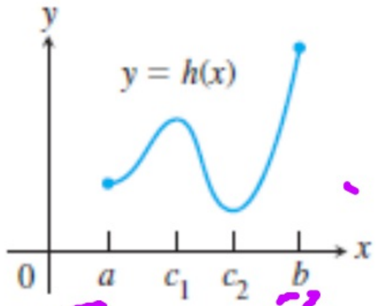
Maximum and minimum at endpoints

Maximum at interior point, minimum at endpoint

Minimum at interior point, maximum at endpoint

In Exercises 5–10, identify each x-value at which any absolute extreme value occurs. Explain how your answer is consistent with the Extreme Value Theorem. See page 195.

5.



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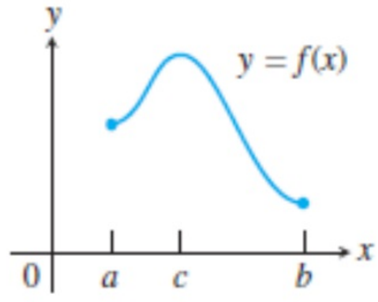

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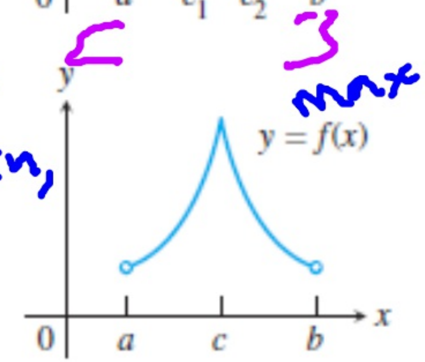

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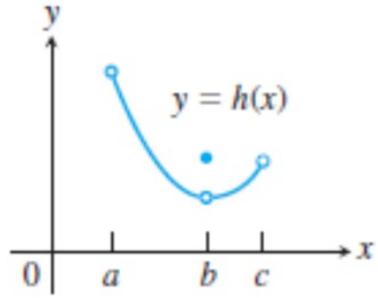

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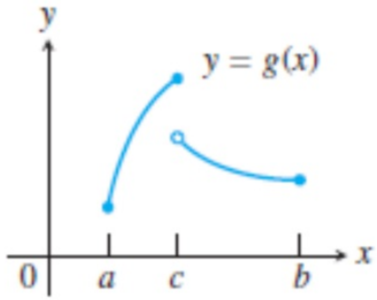

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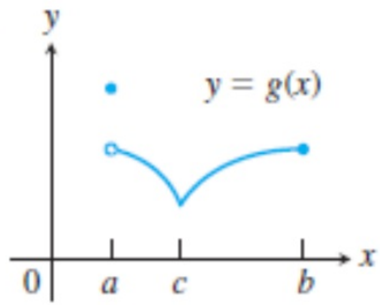


No end pts.

NOT continuous

(8) No max, No min

DEFINITION Local Extreme Values

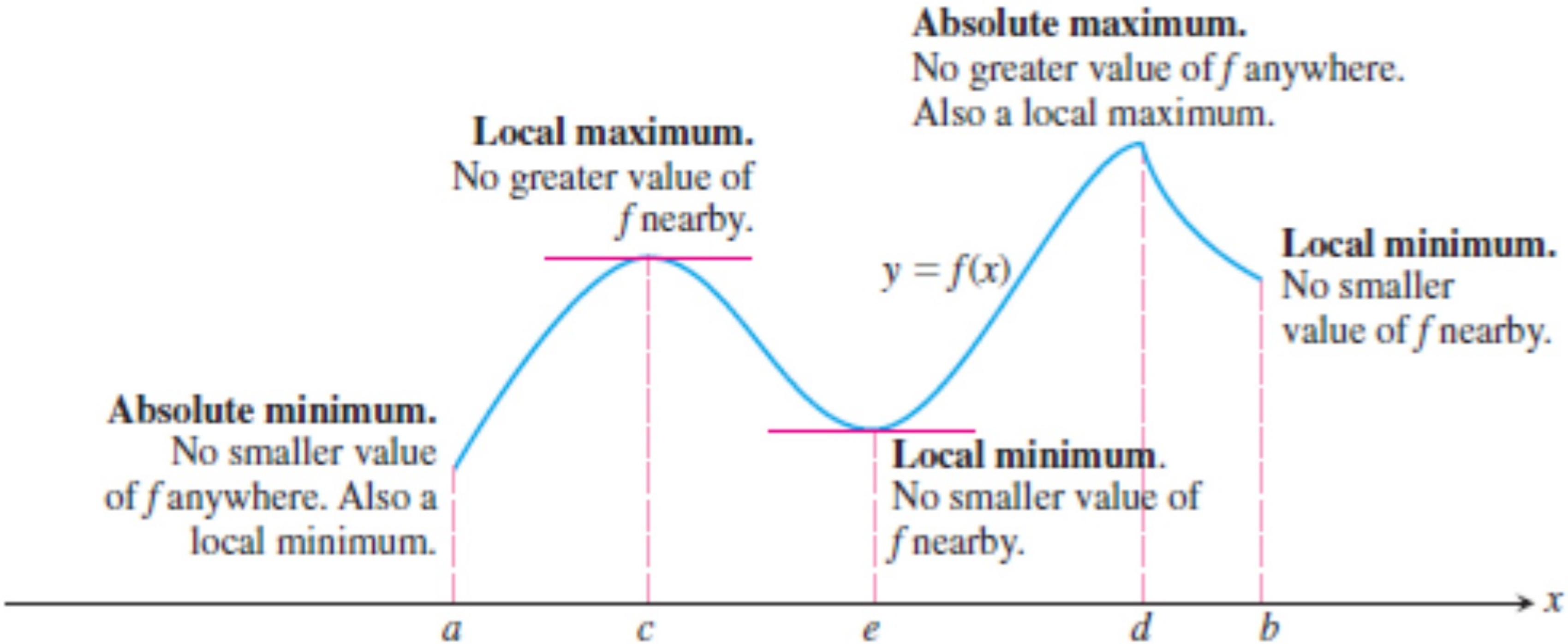
Let c be an interior point of the domain of the function f. Then $f(c)$ is a

(a) **local maximum value** at c if and only if $f(x) \leq f(c)$ for all x in some open interval containing c.

(b) **local minimum value** at c if and only if $f(x) \geq f(c)$ for all x in some open interval containing c.

A function f has a local maximum or local minimum *at an endpoint* c if the appropriate inequality holds for all x in some half-open domain interval containing c.

Note how relative extremas are considered around *open* intervals.



THEOREM 2 Local Extreme Values

If a function f has a local maximum value or a local minimum value at an interior point c of its domain, and if f' exists at c, then

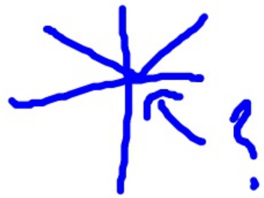
$$f'(c) = 0.$$

From this theorem, we are able to identify *critical points;*

DEFINITION Critical Point

A point in the interior of the domain of a function f at which $f' = 0$ or f' does not exist is a **critical point** of f.

Ex. recall the graph of f(x)=|x|...



EXAMPLE 3 Finding Absolute Extrema

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

closed → Test end points!

SOLUTION

Solve Graphically Figure 4.5 suggests that f has an absolute maximum value of about 2 at $x = 3$ and an absolute minimum value of 0 at $x = 0$.

Confirm Analytically We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

← can't be zero...

has no zeros but is undefined at $x = 0$. The values of f at this one critical point and at the endpoints are

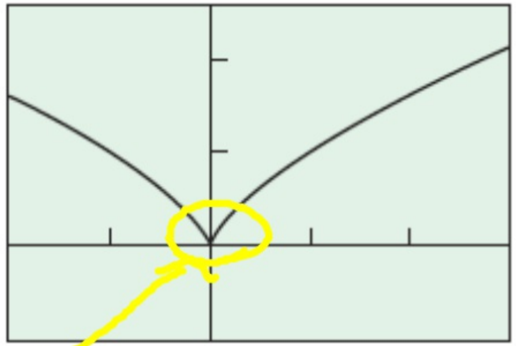
Critical point value: $f(0) = 0$;

Endpoint values: $f(-2) = (-2)^{2/3} = \sqrt[3]{4}$;

$f(3) = (3)^{2/3} = \sqrt[3]{9}$.

We can see from this list that the function's absolute maximum value is $\sqrt[3]{9} \approx 2.08$, and occurs at the right endpoint $x = 3$. The absolute minimum value is 0, and occurs at the interior point $x = 0$. *Now try Exercise 11.*

$y = x^{2/3}$



$[-2, 3]$ by $[-1, 2.5]$

Figure 4.5 (Example 3)

In Exercises 11–18, use analytic methods to find the extreme values of the function on the interval and where they occur. See page 195.

11. $f(x) = \frac{1}{x} + \ln x, \quad 0.5 \leq x \leq 4$

12. $g(x) = e^{-x}, \quad -1 \leq x \leq 1$

13. $h(x) = \ln(x + 1), \quad 0 \leq x \leq 3$

14. $k(x) = e^{-x^2}, \quad -\infty < x < \infty$

15. $f(x) = \sin\left(x + \frac{\pi}{4}\right), \quad 0 \leq x \leq \frac{7\pi}{4}$

16. $g(x) = \sec x, \quad -\frac{\pi}{2} < x < \frac{3\pi}{2}$

17. $f(x) = x^{2/5}, \quad -3 \leq x < 1$

18. $f(x) = x^{3/5}, \quad -2 < x \leq 3$

(13) $h'(x) = \frac{1}{x+1}$

$x \neq -1$

Test $x = 0$ and $x = 3$

$\ln(1) = 0$ Abs. min

$\ln(4) = 1.4$ Abs. max.

$\ln(0) =$ ☹

(18) $f'(x) = \frac{3}{5} x^{-2/5} = \frac{3}{5\sqrt[5]{x^2}}$

$x \neq 0$

$x = 0$;

$f(0) = 0^{3/5} = 0$ Min

$x = 3$;

$f(3) = 3^{3/5} \approx 1.9$ Max

In Exercises 19–30, find the extreme values of the function and where they occur.

19. $y = 2x^2 - 8x + 9$ Min value 1 at $x = 2$

20. $y = x^3 - 2x + 4$

21. $y = x^3 + x^2 - 8x + 5$

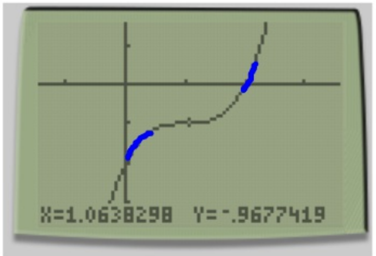
22. $y = x^3 - 3x^2 + 3x - 2$ None ?

$y' = 3x^2 - 6x + 3 = (3x - 3)(x - 1)$

$3(x-1)(x-1)$

$f'(x) = 3(x-1)^2 = 0$

No undefined points



$x = 1$

First Derivative Test

Test $x=0$ Test $x=2$

1

$f'(x) > 0$ $f'(x) > 0$

+ +

No sign change $\Rightarrow$ Not an extrema

In Exercises 11–18, use analytic methods to find the extreme values of the function on the interval and where they occur. See page 195.

11. $f(x) = \dfrac{1}{x} + \ln x, \quad 0.5 \le x \le 4$
12. $g(x) = e^{-x}, \quad -1 \le x \le 1$
13. $h(x) = \ln(x+1), \quad 0 \le x \le 3$
14. $k(x) = e^{-x^2}, \quad -\infty < x < \infty$
15. $f(x) = \sin\left(x + \dfrac{\pi}{4}\right), \quad 0 \le x \le \dfrac{7\pi}{4}$
16. $g(x) = \sec x, \quad -\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$
17. $f(x) = x^{2/5}, \quad -3 \le x < 1$
18. $f(x) = x^{3/5}, \quad -2 < x \le 3$

(13) $\dfrac{d}{dx}[\ln u] \cdot \dfrac{1}{u} \cdot u'$

$u = x+1$

$u' = 1$

$h'(x) = \dfrac{1}{x+1} = 0$

$x \ne -1$

extreme value: $-1, 0, 3$

$h(x) = \ln(x+1)$

$h(-1) = \ln(0)$:(

$h(0) = \ln(1) = 0$ min

$\ln(3) = \ln(4) = 1.4$ max.

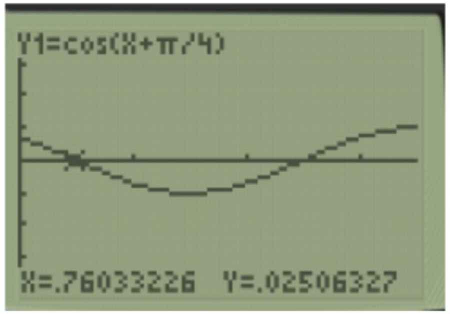
15. $f(x) = \sin\left(x + \frac{\pi}{4}\right), \quad 0 \le x \le \frac{7\pi}{4}$

$\cos\left(x + \frac{\pi}{4}\right) \cdot 1 = 0$

$\cos\left(x + \frac{\pi}{4}\right) = 0$

$u = x + \frac{\pi}{4}$

$u' = 1$



17. $f(x) = x^{2/5}, \quad -3 \le x < 1$

$f'(x) = \frac{2}{5}x^{-3/5} = \frac{2}{5x^{3/5}} = 0$

$x \ne 0 \ldots$

critical #s

$x = 0, -3$

$f(0) = 0^{2/5} = 0$ min

$f(-3) = (-3)^{2/5} = 9^{1/5}$ max

$\frac{2}{x^{3/5}} = 0 \cdot x^{3/5}$

$2 = 0$?