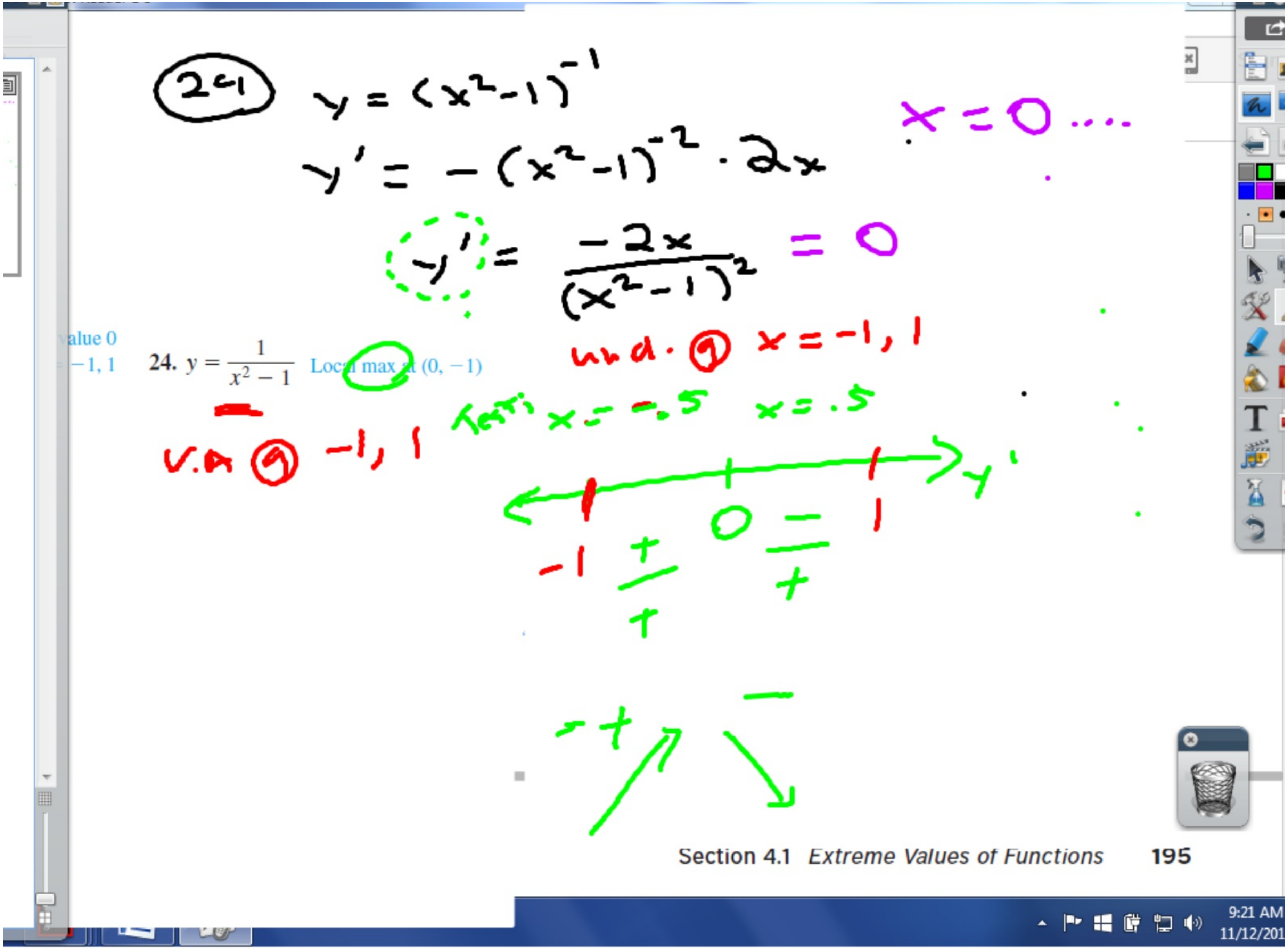


24 $y = (x^2-1)^{-1}$

$y' = -(x^2-1)^{-2} \cdot 2x$

$y' = \dfrac{-2x}{(x^2-1)^2} = 0$

$x = 0$ ....

alue 0

−1, 1

**24.** $y = \dfrac{1}{x^2 - 1}$ Local max at $(0, -1)$

und. @ $x = -1, 1$

V.A @ $-1, 1$

Test $x = -.5$ $x = .5$

$$\xleftarrow{\quad -1 \quad \quad 0 \quad \quad 1 \quad} y'$$

$-1$ $\dfrac{+}{+}$ $0$ $\dfrac{-}{+}$ $1$

$+ \nearrow$ $\overset{-}{\searrow}$

Section 4.1 *Extreme Values of Functions* 195

# 4.2 The Mean Value Theorem

**THEOREM 3** **Mean Value Theorem for Derivatives**

If $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior $(a, b)$, then there is at least one point $c$ in $(a, b)$ at which

Alexia

$$\underset{m}{f'(c)} = \frac{\overset{y_2-y_1}{f(b) - f(a)}}{\underset{x_2-x_1}{b - a}}.$$

me

then

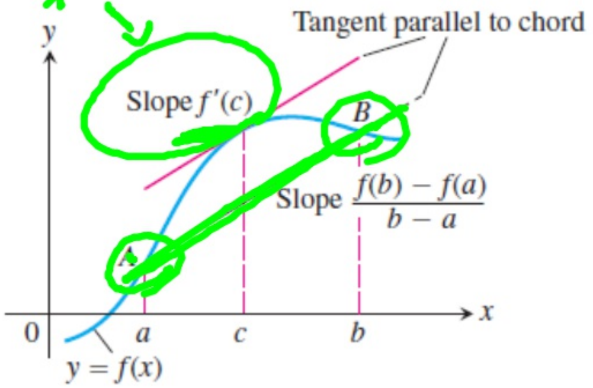
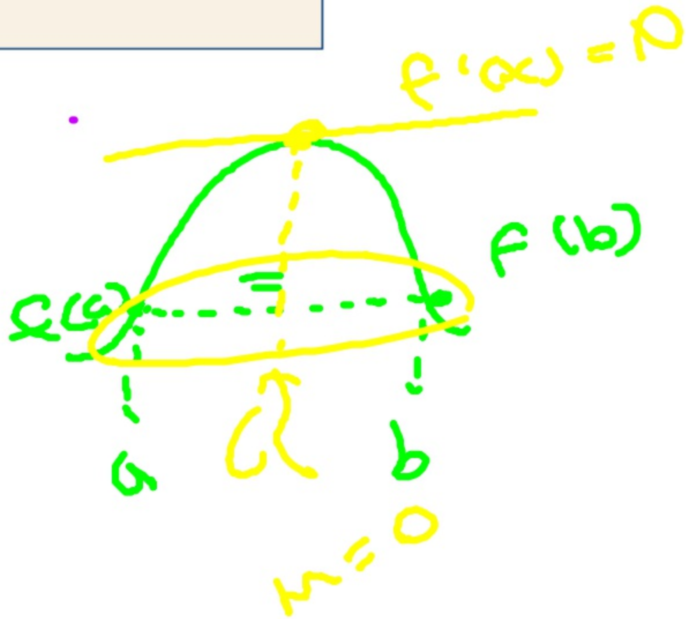


**Figure 4.10** Figure for the Mean Value Theorem.

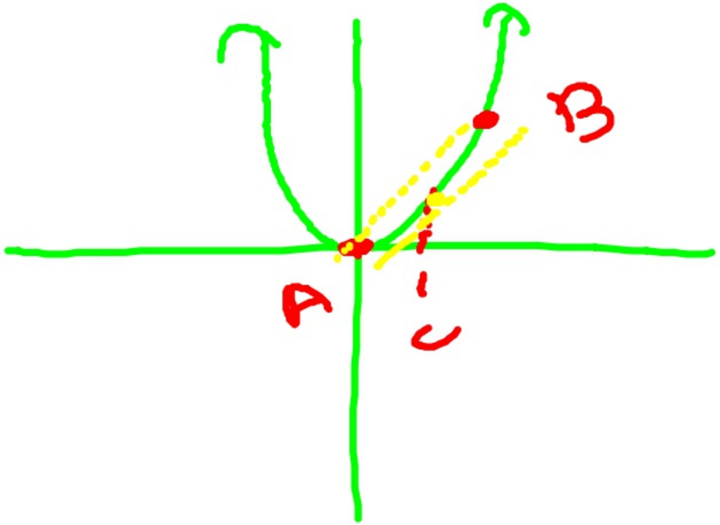


$$c = 1.$$

**nterpret** The tangent line to $f(x) = x^2$ at $x = 1$ has slope 2 and is parallel to t hord joining $A(0, 0)$ and $B(2, 4)$ (Figure 4.12).

**Now try Exer**

*continued*



### THEOREM 3 Mean Value Theorem for Derivatives

If $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior $(a, b)$, then there is at least one point $c$ in $(a, b)$ at which

$$f(b) - f(a)$$

### EXAMPLE 2 Exploring the Mean Value Theorem

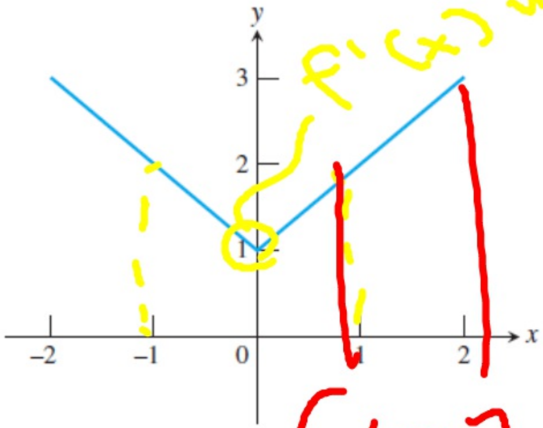
Explain why each of the following functions fails to satisfy the conditions of the Mean Value Theorem on the interval [–1, 1].

[1, 2½]

**(a)** $f(x) = \sqrt{x^2} + 1$ **(b)** $f(x) = \begin{cases} x^3 + 3 & \text{for } x < 1 \\ x^2 + 1 & \text{for } x \geq 1 \end{cases}$

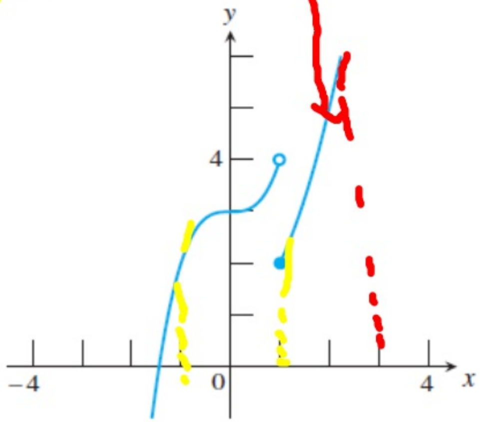
**SOLUTION**

f'(x) und



[1, 2]

(a)



(b)

### THEOREM 3 Mean Value Theorem for Derivatives

If $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior $(a, b)$, then there is at least one point $c$ in $(a, b)$ at which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In Exercises 1–8, **(a)** state whether or not the function satisfies the hypotheses of the Mean Value Theorem on the given interval, and **(b)** if it does, find each value of $c$ in the interval $(a, b)$ that satisfies the equation

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

**1.** $f(x) = x^2 + 2x - 1$ on $[\underset{a}{\overset{a}{0}}, \underset{b}{\overset{b}{1}}]$

**2.** $f(x) = x^{2/3}$ on $[0, 1]$

**3.** $f(x) = x^{1/3}$ on $[-1, 1]$ No. There is a vertical tangent at $x = 0$.

**4.** $f(x) = |x - 1|$ on $[0, 4]$ No. There is a corner at $x = 1$.

**5.** $f(x) = \sin^{-1} x$ on $[-1, 1]$

**6.** $f(x) = \ln(x - 1)$ on $[2, 4]$

**8.** $f(x) = \begin{cases} \sin^{-1} x, & -1 \le x < 1 \\ x/2 + 1, & 1 \le x \le 3 \end{cases}$ on $[-1, 3]$

$f(c) = c^2 + 2c - 1$

$f'(c) = 2c + 2$

$f(1)$

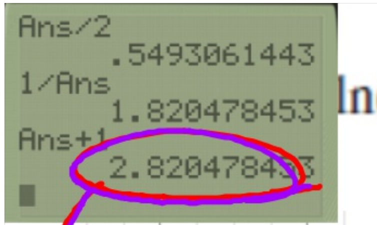
$f(b) = 1 + 2 - 1 = \overset{f(b)}{2}$

$f(a) = f(0) = 0 + 0 - 1 = \overset{f(a)}{-1}$

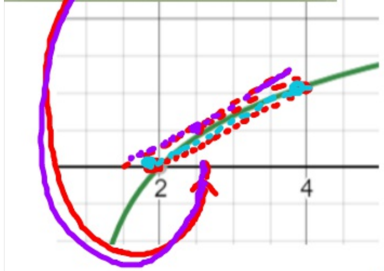
$$2c + 2 = \frac{2 - (-1)}{1 - 0}$$

$$\begin{aligned} 2c + 2 &= 3 \\ -2 &\quad -2 \\ \hline 2c &= 1 \end{aligned}$$

$c = 1/$



```
Ans/2
          .5493061443
1/Ans
          1.820478453
Ans+1
          2.820478453
```



$\ln(x - 1)$ on $[2, 4]$

a b

✓ continuous at [2,4]

✓ diff. at (2,4)

$$\frac{d}{dx}\boxed{\ln(u)} = \frac{1}{u}\cdot u'$$

$f(a) = \ln 1$

$f(b) = \ln 3$

$f(4) = \ln(4-1)$

$u = x - 1$

$u' = 1$

$$f'(c) = \frac{1}{c-1}$$

$$\frac{f(b)-f(a)}{b-a} = \frac{\ln 3 - \ln 1}{2}$$

$$\frac{1}{c-1} = .549$$

$$\frac{1}{.549} = \frac{(c-1)(.549)}{.549}$$

**8.** $f(x) = \begin{cases} \sin^{-1} x, & -1 \le x < 1 \\ x/2+1, & 1 \le x \le 3 \end{cases}$ on $[-1, 3]$

No. The split function is discontinuous at $x = 1$.

In Exercises 1–8, (**a**) state whether or not the function satisfies the hypotheses of the Mean Value Theorem on the given interval, and (**b**) if it does, find each value of $c$ in the interval $(a, b)$ that satisfies the equation

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

**8.** $f(x) = \begin{cases} \sin^{-1} x, & -1 \le x < 1 \\ x/2+1, & 1 \le x \le 3 \end{cases}$ on $[-1, 3]$

No. The split function is discontinuous at $x = 1$.

**1.** $f(x) = x^2 + 2x - 1$ on $[\underset{a}{0}, \underset{b}{1}]$

**2.** $f(x) = x^{2/3}$ on $[0, 1]$

**3.** $f(x) = x^{1/3}$ on $[-1,1]$

**4.** $f(x) = |x - 1|$ on $[0, 4]$ No. T

**5.** $f(x) = \sin^{-1} x$ on $[-1, 1]$

**6.** $f(x) = \ln(x - 1)$ on $[2, 4]$

① continuous? ✓

differentiable? ✓

$[0, 1]$

$f'(c) = 2c + 2$

$f(1) = 1^2 + 2(1) - 1$
$= 1 + 2 - 1$
$= 2$

$f(0) = 0^2 + 2(0) - 1$
$= -1$

$f(1) = 2$

$f(0) = -1$

$$\frac{f(1) - f(0)}{1 - 0}$$

$$2c + 2 = \frac{2 - (-1)}{1 - 0}$$

$$2c + 2 = \frac{3}{1}$$

$$2c + 2 = 3$$

$$2c = 1$$

$$\boxed{c = \frac{1}{2}}$$

### EXAMPLE 3 Applying the Mean Value Theorem

Let $f(x) = \sqrt{1 - x^2}$, $A = (-1, f(-1))$, and $B = (1, f(1))$. Find a tangent to $f$ in the interval $(-1, 1)$ that is parallel to the secant $AB$.

**SOLUTION**

The function $f$ (Figure 4.14) is continuous on the interval [–1, 1] and

$$f'(x) = \frac{-x}{\sqrt{1 - x^2}}$$

is defined on the interval $(-1, 1)$. The function is not differentiable at $x = -1$ and $x = 1$, but it does not need to be for the theorem to apply. Since $f(-1) = f(1) = 0$, the tangent we are looking for is horizontal. We find that $f' = 0$ at $x = 0$, where the graph has the horizontal tangent $y = 1$. *Now try Exercise 9.*

### COROLLARY 2 Functions with $f' = 0$ are Constant

If $f'(x) = 0$ at each point of an interval $I$, then there is a constant $C$ for which $f(x) = C$ for all $x$ in $I$.

In Exercises 9 and 10, the interval $a \le x \le b$ is given. Let $A = (a, f(a))$ and $B = (b, f(b))$. Write an equation for

(a) the secant line $AB$.

(b) a tangent line to $f$ in the interval $(a, b)$ that is parallel to $AB$.

**9.** $f(x) = x + \dfrac{1}{x}$, $0.5 \le x \le 2$ **(a)** $y = \dfrac{5}{2}$ **(b)** $y = 2$

**10.** $f(x) = \sqrt{x - 1}$, $1 \le x \le 3$

$\frac{1}{.5}$

$\frac{10}{5} = ?$

$f(b) = f(2) = 2 + \frac{1}{2}$

$f(a) = f(.5) = .5 + 2$

$f(b) - f(a) =$

### COROLLARY 1 Increasing and Decreasing Functions

Let $f$ be continuous on $[a, b]$ and differentiable on $(a, b)$.

1. If $f' > 0$ at each point of $(a, b)$, then $f$ increases on $[a, b]$.
2. If $f' < 0$ at each point of $(a, b)$, then $f$ decreases on $[a, b]$.

### EXAMPLE 6 Determining Where Graphs Rise or Fall

Where is the function $\underline{f(x) = x^3 - 4x}$ increasing and where is it decreasing?

**SOLUTION**

**Solve Graphically** The graph of $f$ in Figure 4.17 suggests that $f$ is increasing from $-\infty$ to the $x$-coordinate of the local maximum, decreasing between the two local extrema, and increasing again from the $x$-coordinate of the local minimum to $\infty$. This information is supported by the superimposed graph of $f'(x) = \underline{3x^2 - 4}$.

**Confirm Analytically** The function is increasing where $f'(x) > 0$.

$$3x^2 - 4 > 0$$

$$x^2 > \frac{4}{3}$$

$$x < -\sqrt{\frac{4}{3}} \quad \text{or} \quad x > \sqrt{\frac{4}{3}}$$

skip
#11-14!

The function is decreasing where $f'(x) < 0$.

$$3x^2 - 4 < 0$$

$$x^2 < \frac{4}{3}$$

$$-\sqrt{\frac{4}{3}} < x < \sqrt{\frac{4}{3}}$$

In Exercises 15–22, use analytic methods to find **(a)** the local extrema, **(b)** the intervals on which the function is increasing, and **(c)** the intervals on which the function is decreasing.

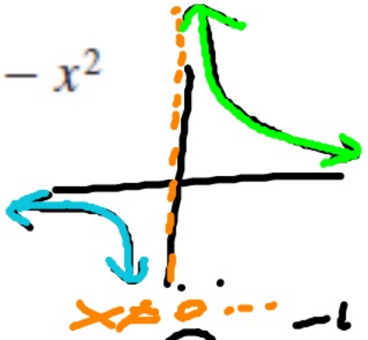
First Derivative Test...

**15.** $f(x) = 5x - x^2$

**16.** $g(x) = x^2 - x - 12$

**17.** $h(x) = \frac{2}{x}$

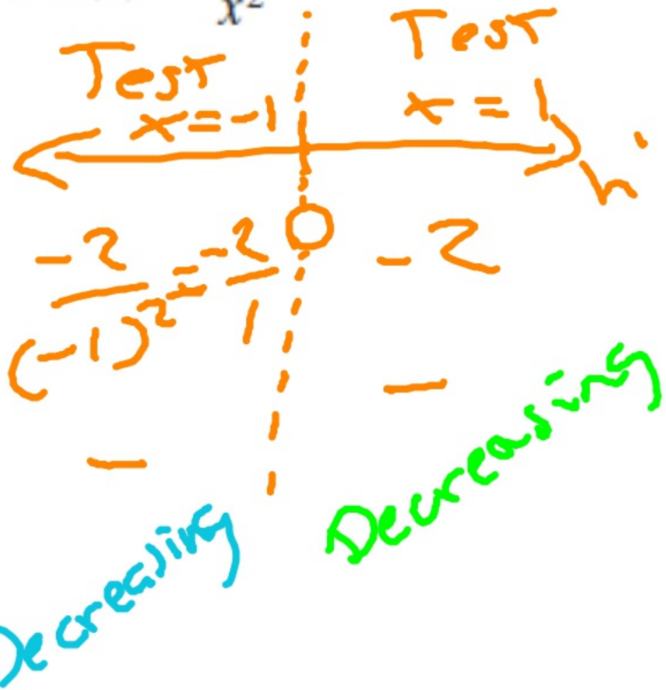
**18.** $k(x) = \frac{1}{x^2}$



⑰ $h(x) = 2x^{-1}$

$h'(x) = -2x^{-2}$

$h'(x) = -\frac{2}{x^2}$



$\frac{-2}{(-1)^2} = \frac{-2}{1}$

$-$

$\frac{-2}{}$

$-$

In Exercises 15–22, use analytic methods to find **(a)** the local extrema, **(b)** the intervals on which the function is increasing, and **(c)** the intervals on which the function is decreasing.

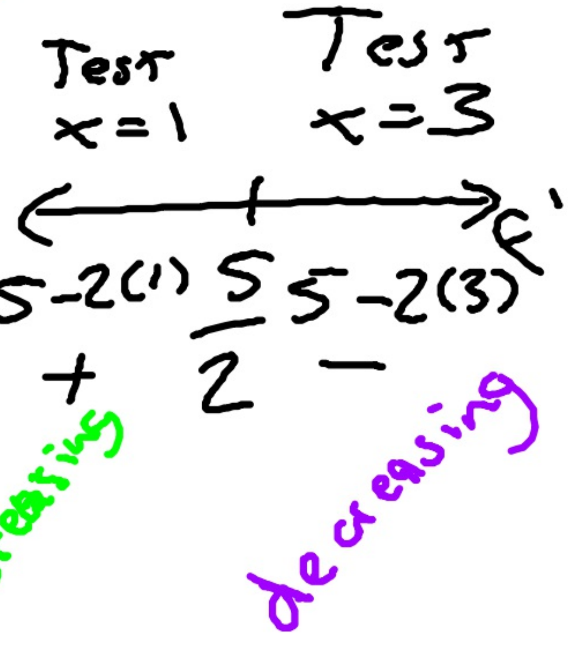
**15.** $f(x) = 5x - x^2$ See page 204. **16.** $g(x) = x^2 - x - 12$ See page 204.

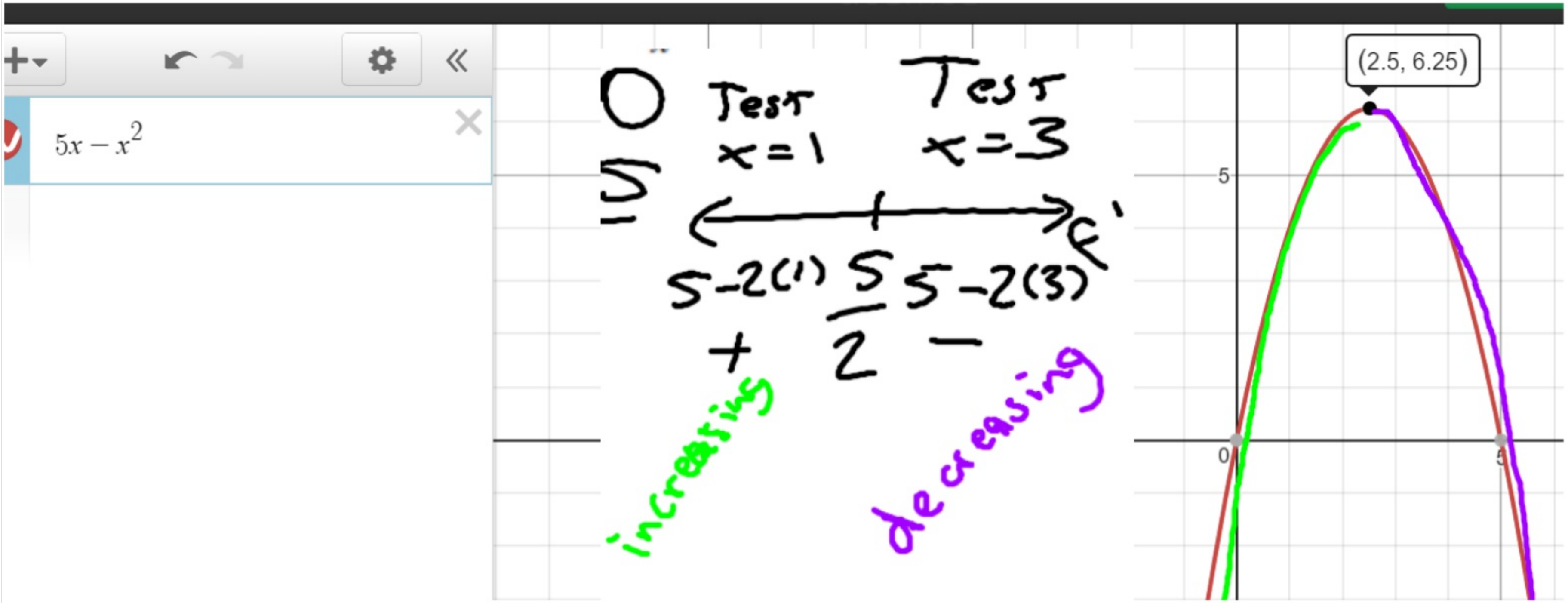
**17.** $h(x) = \frac{2}{x}$ See page 204. **18.** $k(x) = \frac{1}{x^2}$ See page 204.

⑮ $f'(x) = 5 - 2x = 0$

$$\begin{aligned} -5 \quad & \quad -5 \\ \hline \frac{-2x}{-2} &= \frac{-5}{-2} \end{aligned}$$

$x = 5/2$





$5x - x^2$

In Exercises 23–28, find (**a**) the local extrema, (**b**) the intervals on which the function is increasing, and (**c**) the intervals on which the function is decreasing.

**23.** $f(x) = x\sqrt{4 - x}$

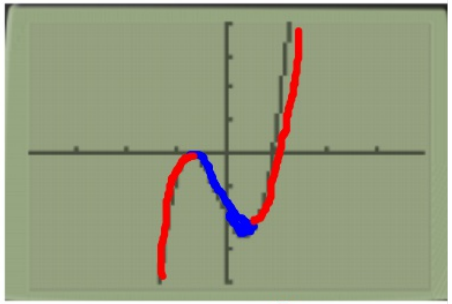
**24.** $g(x) = x^{1/3}(x + 8)$

**25.** $h(x) = \dfrac{-x}{x^2 + 4}$

**26.** $k(x) = \dfrac{x}{x^2 - 4}$

**27.** $f(x) = x^3 - 2x - 2\cos x$

**28.** $g(x) = 2x + \cos x$



graphed it ↑ f

∴

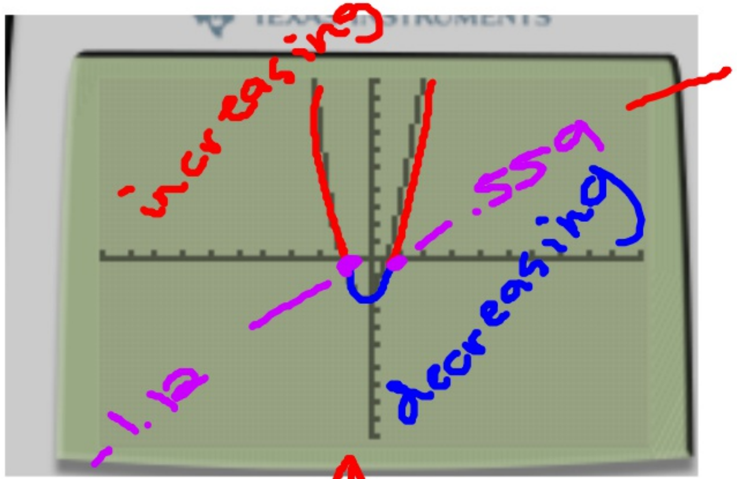
㉗ $\overset{Y_1}{f'(x)} = 3x^2 - 2 + 2\sin x = 0$

increasing:

$(-\infty, -1.12) \cup (.559, \infty)$

decreasing:

$(-1.12, .559)$



f'

In Exercises 23–28, find **(a)** the local extrema, **(b)** the intervals on which the function is increasing, and **(c)** the intervals on which the function is decreasing.

**23.** $f(x) = x\sqrt{4-x}$

**24.** $g(x) = x^{1/3}(x+8)$

**24. (a)** Local min at $\approx (-2, -7.56)$ **(b)** On $[-2, \infty)$ **(c)** On $(-\infty, -2]$

**25.** $h(x) = \dfrac{-x}{x^2+4}$ $\begin{matrix}\leftarrow f\\ \leftarrow g\end{matrix}$

**26.** $k(x) = \dfrac{x}{x^2-4}$

**27.** $f(x) = x^3 - 2x - 2\cos x$

**28.** $g(x) = 2x + \cos x$

**(a)** None **(b)** On $(-\infty, \infty)$ **(c)** None

㉕ $f = -x$ $\quad g = x^2+4$

$f' = -1$ $\quad g' = 2x$

$-x^2 - 4 - (-2x^2)$

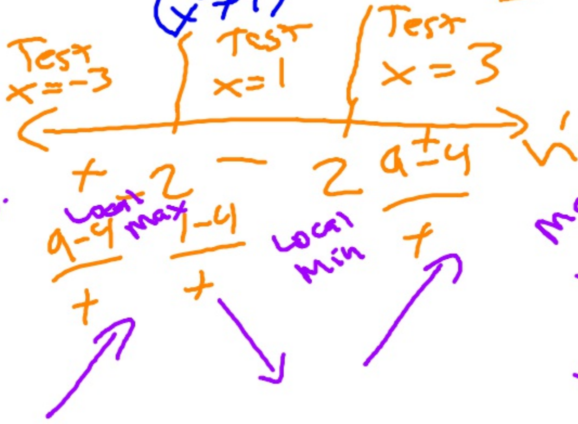
$$h'(x) = \frac{(x^2+4)(-1) - (-x)(2x)}{(x^2+4)^2}$$

$x^2 - 4 = 0$

$$h'(x) = \frac{x^2-4}{(x^2+4)^2} = 0$$

$(x+2)(x-2) = 0$

$\underline{x = -2, 2}$



Increasing: $(-\infty, -2) \cup (2, \infty)$

Decreasing: $(-2, 2)$

Max @ $x = -2$

Min @ $x = 2$

**COROLLARY 3** **Functions with the Same Derivative Differ by a Constant**

If $f'(x) = g'(x)$ at each point of an interval $I$, then there is a constant $C$ such that $f(x) = g(x) + C$ for all $x$ in $I$.

$\boxed{?,}$

$x^n$

$n \cdot x^{n-1}$

**DEFINITION** **Antiderivative**

A function $F(x)$ is an **antiderivative** of a function $f(x)$ if $F'(x) = f(x)$ for all $x$ in the domain of $f$. The process of finding an antiderivative is **antidifferentiation.**

$F(x) =$

$f(x) =$

$f'(x) =$

$f''(x)$

## Think of deriving, only in reverse....

In Exercises 29–34, find all possible functions $f$ with the given derivative.

$f'(x) = 2x^0$

**29.** $f'(x) = x$ $\quad \frac{x^2}{2} + C$

**30.** $f'(x) = 2$ $\quad 2x + C$

**31.** $f'(x) = 3x^2 - 2x + 1$ $\quad x^3 - x^2 + x + C$

**32.** $f'(x) = \sin x$ $\quad -\cos x + C$

**33.** $f'(x) = e^x$ $\quad e^x + C$

**34.** $f'(x) = \frac{1}{x-1}, \quad x > 1$ $\quad \ln(x-1) + C$

$\boxed{29}$ $\frac{1}{2} x^2$

### COROLLARY 3 Functions with the Same Derivative Differ by a Constant

If $f'(x) = g'(x)$ at each point of an interval $I$, then there is a constant $C$ such that $f(x) = g(x) + C$ for all $x$ in $I$.

Ex. $f(x) = x^2 + 2$

$g(x) = x^2 + 0$

$g'(x) = 2x$

$f'(x) = 2x$

## EXAMPLE 7 Applying Corollary 3

Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0, 2)$.

x y

**SOLUTION**

Since $f$ has the same derivative as $g(x) = -\cos x$, we know that $f(x) = -\cos x + C$, for some constant $C$. To identify $C$, we use the condition that the graph must pass through $(0, 2)$. This is equivalent to saying that

$$\begin{aligned} \underline{f(0)} &= \underline{2} \\ -\cos(0) + C &= 2 && f(x) = -\cos x + C \\ -1 + C &= 2 \\ C &= 3. \end{aligned}$$

The formula for $f$ is $f(x) = -\cos x + 3$.

***Now try Exercise 35.***

In Exercises 35–38, find the function with the given derivative whose graph passes through the point $P$.

**35.** $f'(x) = -\dfrac{1}{x^2}, \quad x > 0, \quad P(2, 1)$

$x$ $f(x)$

**36.** $f'(x) = \dfrac{1}{4x^{3/4}}, \quad P(1, -2)$

**37.** $f'(x) = \dfrac{1}{x + 2}, \quad x > -2, \quad P(-1, 3)$

**38.** $f'(x) = 2x + 1 - \cos x, \quad P(0, 3)$

35

$f'(x) = -x^{-2}$

$f(x) = x^{-1}$

$f(x) = -1/x + -1/2$

$$f(x) = x^{-1} + C$$

$$1 = 2^{-1} + C$$

$$1 = \frac{1}{2} + C$$

$$-\frac{1}{2} \quad -\frac{1}{2}$$

$$C = \frac{1}{2}$$

**36.** $f'(x) = \dfrac{1}{4x^{3/4}}$, $P(1, -2)$ $x^{1/4} - 3$

$x$ $f(x)$

**37.** $f'(x) = \dfrac{1}{x+2}$, $x > -2$, $P(-1, 3)$ $\ln(x+2) + 3$

**38.** $f'(x) = 2x + 1 - \cos x$, $P(0, 3)$ $x^2 + x - \sin x + 3$

$$f'(x) = \frac{1}{4}x^{-3/4}$$

$$f(x) = x^{1/4} + C$$

$$-2 = 1^{1/4} + C$$

$$-2 = 1 + C$$

$$C = -3$$

$$f(x) = x^{1/4} - 3$$

**Group Activity** In Exercises 39–42, sketch a graph of a differentiable function $y = f(x)$ that has the given properties.

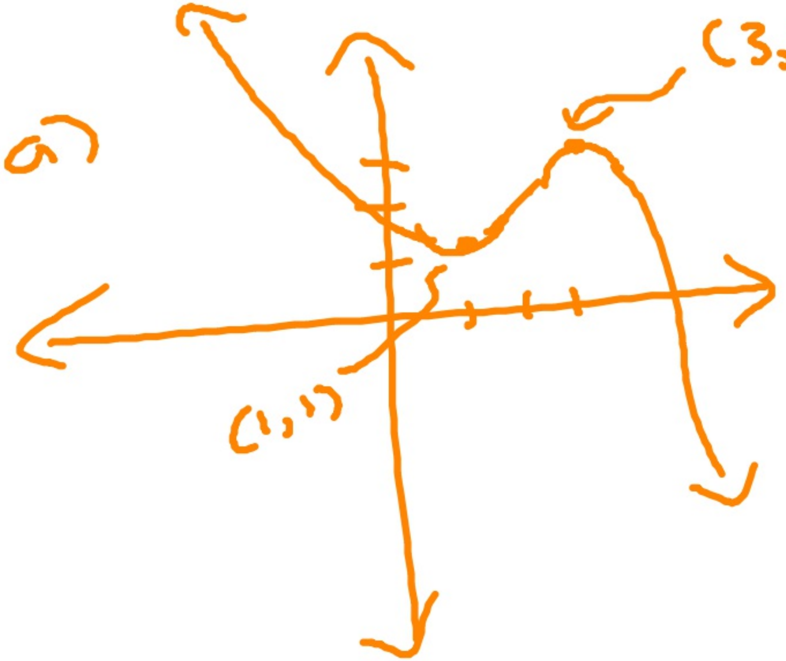
**39.** **(a)** local minimum at (1, 1), local maximum at (3, 3)

**(b)** local minima at (1, 1) and (3, 3)

**(c)** local maxima at (1, 1) and (3, 3)

39

a)



b)

**40.** $f(2) = 3,\quad f'(2) = 0,\quad$ and

(a) $f'(x) > 0$ for $x < 2$, $\quad f'(x) < 0$ for $x > 2$.

(b) $f'(x) < 0$ for $x < 2$, $\quad f'(x) > 0$ for $x > 2$.

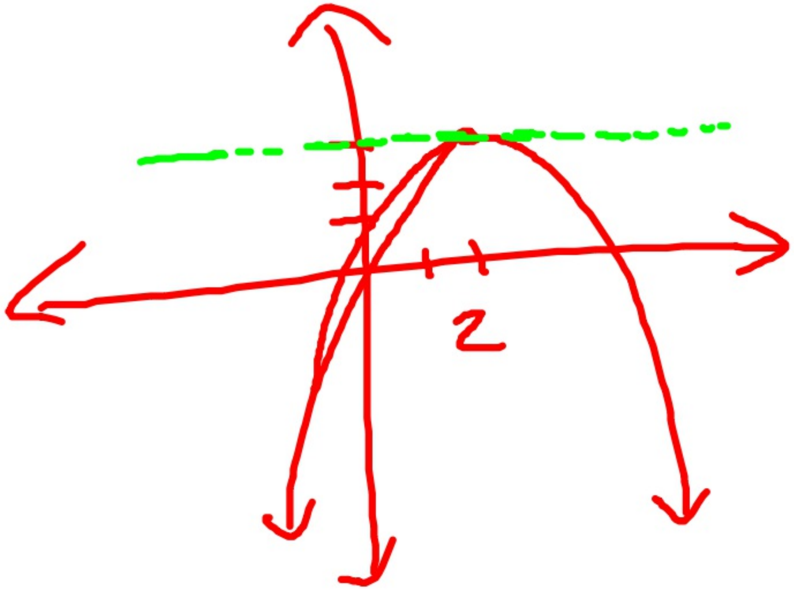
(c) $f'(x) < 0$ for $x \neq 2$.

(d) $f'(x) > 0$ for $x \neq 2$.

positive slope

negative slope

40



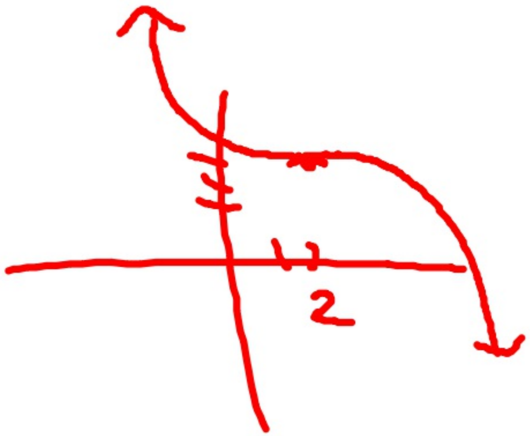
**(b)** $f'(x) < 0$ for $x < 2$, $f'(x) > 0$ for $x > 2$.

**(c)** $f'(x) < 0$ for $x \neq 2$.

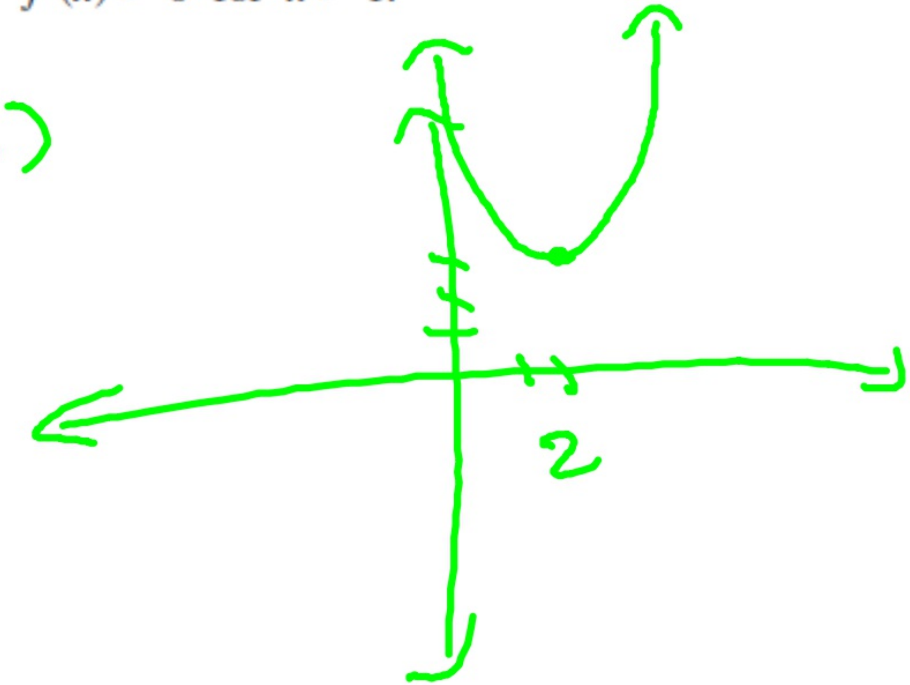
**(d)** $f'(x) > 0$ for $x \neq 2$.

**41.** $f'(-1) = f'(1) = 0$, $f'(x) > 0$ on $(-1, 1)$,
$f'(x) < 0$ for $x < -1$, $f'(x) > 0$ for $x > 1$.

c)



b)



**41.** $f'(-1) = f'(1) = 0$, $f'(x) > 0$ on $(-1, 1)$, ← interval notation!
$f'(x) < 0$ for $x < -1$, $f'(x) > 0$ for $x > 1$.