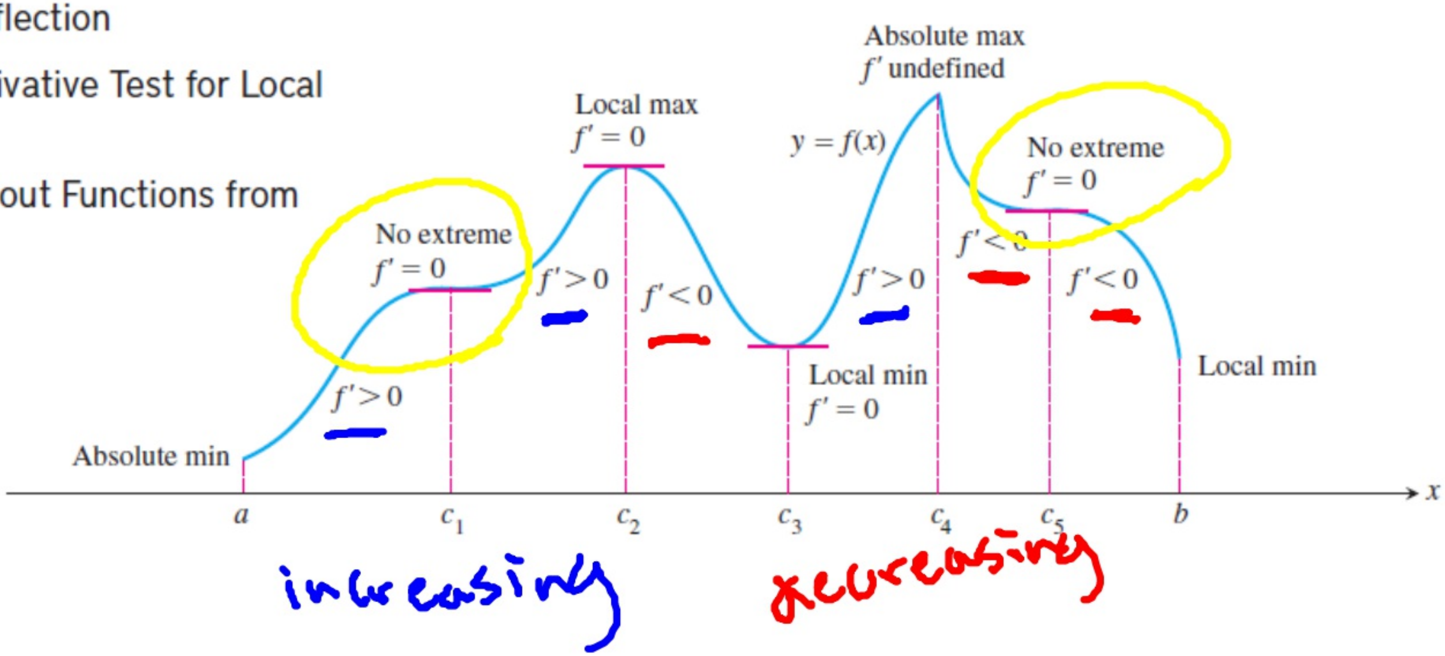
4.3 Connecting f' and f'' with the Graph of f

What you'll learn about

- First Derivative Test for Local Extrema
- Concavity
- Points of Inflection
- Second Derivative Test for Local Extrema
- Learning about Functions from Derivatives

We will use what we know about derivatives to identify extremas;

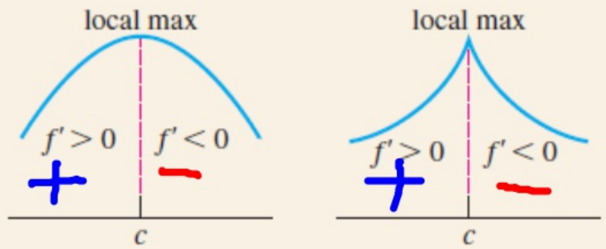


THEOREM 4 First Derivative Test for Local Extrema

The following test applies to a continuous function $f(x)$.

At a critical point c:

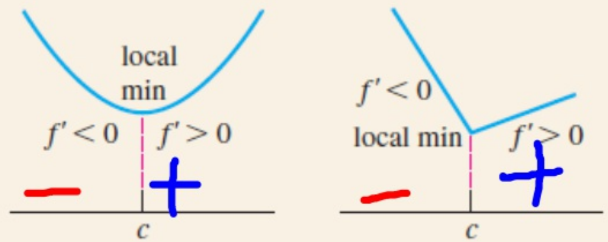
1. If f' changes sign from positive to negative at c ($f' > 0$ for $x < c$ and $f' < 0$ for $x > c$), then f has a local maximum value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

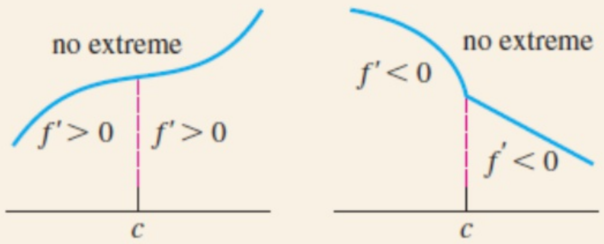
notice how the sign of f' changes around extremas!

2. If f' changes sign from negative to positive at c ($f' < 0$ for $x < c$ and $f' > 0$ for $x > c$), then f has a local minimum value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

3. If f' does not change sign at c (f' has the same sign on both sides of c), then f has no local extreme value at c.

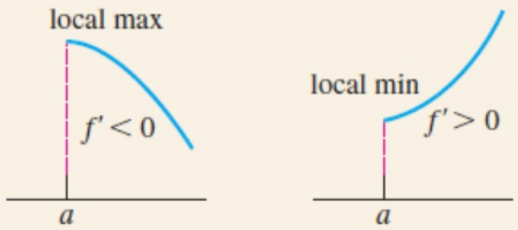


(a) $f'(c) = 0$ (b) $f'(c)$ undefined

Also, note how extremas don't occur whenever the sign of f' does NOT change...

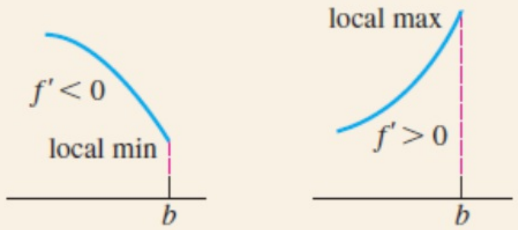
At a left endpoint a:

If $f' < 0$ ($f' > 0$) for $x > a$, then f has a local maximum (minimum) value at a.



At a right endpoint b:

If $f' < 0$ ($f' > 0$) for $x < b$, then f has a local minimum (maximum) value at b.



EXAMPLE 1 Using the First Derivative Test

For each of the following functions, use the First Derivative Test to find the local extreme values. Identify any absolute extrema.

(a) $f(x) = x^3 - 12x - 5$

$f'(x) = 3x^2 - 12$

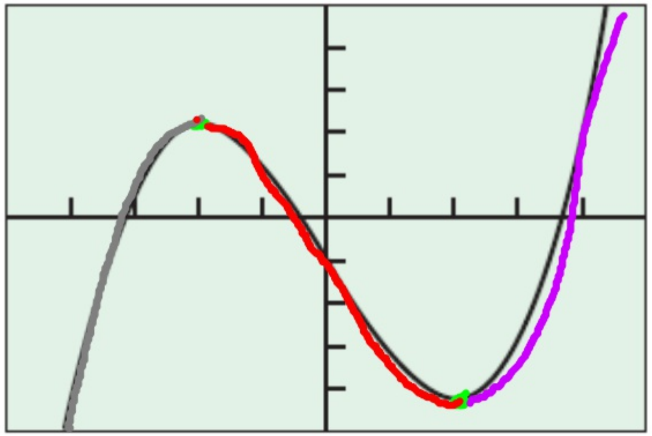
$3x^2 - 12 = 0$

$+12 \quad +12$

$\frac{3x^2}{3} = \frac{12}{3}$

$\sqrt{x^2} = \sqrt{4}$

$x = \pm 2$



$3(0)^2 - 12$

$= -12$

$3(4)^2 - 12$

$3(16)$

$48 - 12 = 36$

SOLUTION

(a) Since f is differentiable for all real numbers, the only possible critical points are the zeros of f'. Solving $f'(x) = 3x^2 - 12 = 0$, we find the zeros to be $x = 2$ and $x = -2$. The zeros partition the x-axis into three intervals, as shown below:

Sign of f' +   −   +

Test x...   Test x=0   Test x=4

-2 2 x

4. $y = xe^{1/x}$ Local minimum: $(1, e)$

$x=1 \ldots \quad y = (1)e^{1/1}$

$\frac{d}{dx} e^u = e^u \cdot u'$

$u = -x$

$u' = -1$

4 $\quad y = x \cdot e^{-x}$

$f = x \qquad g = e^{-x}$

$f' = 1 \qquad g' = -e^{-x}$

$f' = (1)(e^{-x}) + (-e^{-x})(x)$

$f' = e^{-x} - e^{-x} \cdot x = 0$

GCF: e^{-x}

$e^{-2} = \frac{1}{e^2}$

$+$

$f' = e^{-x}(1-x) = 0$

zero!! $\quad e^{-x} = 0 \qquad 1 - x = 0$

$\frac{1}{e^x} = 0 \qquad x = 1$

$x=0 \qquad x = 2$

$e^0(1-0) \mid e^{-2}(1-2)$

$x=1 \qquad + \qquad -$

Local maxima: $(-\sqrt{8}, 0)$ and $(2, 4)$; local minima: $(-2, -4)$ and $(\sqrt{8}, 0)$; 4 is an absolute maximum and -4 is an absolute minimum.

5. $y = x\sqrt{8 - x^2}$

Local minimum: $(0, 1)$

6. $y = \begin{cases} 3 - x^2, & x < 0 \\ x^2 + 1, & x \geq 0 \end{cases}$

5 $\quad y = x(8-x^2)^{1/2}$

$f = x \qquad g = (8-x^2)^{1/2}$

$f' = 1 \qquad g' = \frac{1}{2}(8-x^2)^{-1/2}(-2x)$

$x^m \cdot x^n = x^{m+n}$

$$\frac{2(8-x^2)^{1/2}}{2(8-x^2)^{1/2}} \cdot \frac{(8-x^2)^{1/2}}{1} + \frac{-2x^2}{2(8-x^2)^{1/2}} = 0$$

$$\frac{2(8-x^2) + (-2x^2)}{2(8-x^2)^{1/2}} = \frac{16-2x^2-2x^2}{2(8-x^2)^{1/2}}$$

$$= \frac{\cancel{16}^{8} - \cancel{4}^{2}x^2}{\cancel{2}(8-x^2)^{1/2}} = \frac{8-2x^2}{(8-x^2)^{1/2}} = 0$$

corners/cusp: $8 - x^2 = 0$, $+x^2 \quad +x^2$, $8 = x^2$, $x = \pm\sqrt{8} = \pm 2\sqrt{2}$

max/min: $8 - 2x^2 = 0$

$8 = 2x^2$

$x^2 = 4$

$x = \pm 2$

In Exercises 1–6, use the First Derivative Test to determine the local extreme values of the function, and identify any absolute extrema. Support your answers graphically.

1. $y = x^2 - x - 1$ **2.** $y = -2x^3 + 6x^2 - 3$

① $y = x^2 - x - 1$

$y' = 2x - 1 = 0$

$+1 \quad +1$

$2x = 1$

$\frac{2x}{2} = \frac{1}{2}$

$x = \frac{1}{2}$

① Derive, set equal to zero. Find x!

$y = \left(\frac{1}{2}\right)^2 - \frac{1}{2} - 1$

$\frac{1}{4} - \frac{1}{2} - 1$

$= -\frac{1}{4} - 1 = -1\frac{1}{4}$

Test $x = 0$ Test $x = 1$

f'

$\frac{1}{2}$

$2(0) - 1$ $2(1) - 1$

$-$ $+$

decreases increases

② Test around crit #

Find it!

min

$\left(\frac{1}{2}, -1\frac{1}{4}\right)$

③ Use f to locate extrema

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1. $y = x^2 - x - 1$

2. $y = -2x^3 + 6x^2 - 3$

3. $y = 2x^4 - 4x^2 + 1$

4. $y = xe^{1/x}$ Local minimum: $(1, e)$

Local maxima: $(-\sqrt{8}, 0)$ and $(2, 4)$;

5. $y = x\sqrt{8 - x^2}$

local minima: $(-2, -4)$ and $(\sqrt{8}, 0)$;

4 is an absolute maximum and -4 is an absolute minimum.

Local minimum: $(0, 1)$

6. $y = \begin{cases} 3 - x^2, & x < 0 \\ x^2 + 1, & x \geq 0 \end{cases}$

(2) $y' = -6x^2 + 12x = 0 \quad x = 0, 2$

$-6x(x-2) = 0$

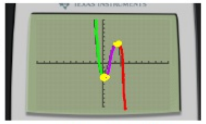
Test $x=-1$ Test $x=1$ Test $x=3$

f'

$(+)(-)$ 0 $(-)(-)$ 2 $(-)(+)$

$-$ $+$ $-$

min max



(5) $y = x\sqrt{8-x^2}$

f g

$8 - x^2 > 0$

$8 > x^2$

$x^2 < 8$

$x < \sqrt{8}$ $x > -\sqrt{8}$

$(-\sqrt{8}, \sqrt{8})$

$f = x$ $\quad g = (8-x^2)^{\frac{1}{2}}$

$f' = 1$ $\quad g' = \frac{1}{2}(8-x^2)^{-\frac{1}{2}}(-2x)$

$= -x(8-x^2)^{-\frac{1}{2}}$

$y' = (1)(8-x^2)^{\frac{1}{2}} - x^2(8-x^2)^{-\frac{1}{2}}$

$= \frac{\sqrt{8-x^2}}{1} - \frac{x^2}{\sqrt{8-x^2}} = \frac{8-x^2}{\sqrt{8-x^2}} - \frac{x^2}{\sqrt{8-x^2}}$

$y' = \frac{8-2x^2}{\sqrt{8-x^2}} = 0$

$8 - 2x^2 = 0$

$-8 \quad -8$

$-2x^2 = -8$

$\sqrt{x^2} = \sqrt{4}$

$x = \pm 2$

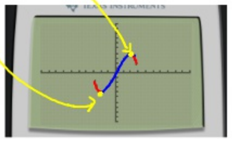
Test $x=0$

y'

$-\sqrt{8}$ -2 $+2$ $+\sqrt{8}$

$-$ $+$ $-$

min max



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3. $y = 2x^4 - 4x^2 + 1$

4. $y = xe^{1/x}$ Local minimum: $(1, e)$

Local maxima: $(-\sqrt{8}, 0)$ and $(2, 4)$;

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local minima: $(-2, -4)$ and $(\sqrt{8}, 0)$;

4 is an absolute maximum and -4 is an absolute minimum.

6. $y = \begin{cases} 3 - x^2, & x < 0 \\ x^2 + 1, & x \ge 0 \end{cases}$

(4) $y = x e^{1/x}$

$f \uparrow \quad g \uparrow$

$f = x \quad g = e^{x^{-1}}$

$f' = 1 \quad g' = e^{\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right)$

$y' = e^{\frac{1}{x}} + x \cdot e^{\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right)$

$y' = e^{\frac{1}{x}} - \frac{1}{x} e^{\frac{1}{x}}$

$x \neq 0$

If $x = 1$,

$y' = 0$

$\sqrt{e} - \frac{1}{2}\sqrt{e}$

$= \frac{1}{2}\sqrt{e}$

$x = -1 \ldots$

$(+) + (+)(+)$

$x = \frac{1}{2}$

$x = 2$

$\frac{1}{e} + \frac{1}{e}$

y'

$+ \quad 0 \quad - \quad 1 \quad +$

$e^2 - 2e^2 = -e^2$