4.4 Modeling and Optimization

Suppose I gave you a 12 inch string to make a rectangle.

$\frac{2x}{2} + \frac{2y}{2} = \frac{12}{2}$ ← secondary

$x + y = 6$

$-x \qquad -x$

$y = 6 - x$

xy = biggest Area

② change "primary" into one variable

$x(6-x)$ = biggest Area

$-x^2 + 6x = $ ↗ Area

primary

① find "primary" and "secondary"

③ find extremas (derive, set = to zero)

$y = 6 - 3 = 3$

$A(x) = -x^2 + 6x$

$A'(x) = -2x + 6 = 0$

$x = 3$

$A(3) = -3^2 + 6(3)$

$= -9 + 18$

$A(3) = 9$

④ Answer the question.

what's the biggest area you can make with it?

Strategy for Solving Max-Min Problems

1. **Understand the Problem** Read the problem carefully. Identify the information you need to solve the problem.
2. **Develop a Mathematical Model of the Problem** Draw pictures and label the parts that are important to the problem. Introduce a variable to represent the quantity to be maximized or minimized. Using that variable, write a function whose extreme value gives the information sought.
3. **Graph the Function** Find the domain of the function. Determine what values of the variable make sense in the problem.
4. **Identify the Critical Points and Endpoints** Find where the derivative is zero or fails to exist.
5. **Solve the Mathematical Model** If unsure of the result, support or confirm your solution with another method.
6. **Interpret the Solution** Translate your mathematical result into the problem setting and decide whether the result makes sense.

In Exercises 1–10, solve the problem analytically. Support your answer graphically.

1. ***Finding Numbers*** The sum of two nonnegative numbers is 20. Find the numbers if

   (a) the sum of their squares is as large as possible; as small as possible.

   (b) one number plus the square root of the other is as large as possible; as small as possible.

$x + y = 20$

$f' = 0$

a) $x^2 + y^2 =$ large as possible

b) $x + \sqrt{y} =$ large as possible

a) $x + y = 20$

$y = 20 - x$

$x^2 + y^2 =$ large

$x^2 + (20-x)^2 = \#$

$x^2 + 400 - 40x + x^2 = \#$

$400 - 40x + 2x^2 = \#$

$-40 + 4x = 0$

$x = 10$

$y = 10$

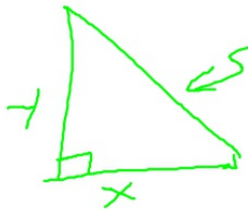
smallest

2. ***Maximizing Area*** What is the largest possible area for a right triangle whose hypotenuse is 5 cm long, and what are its dimensions?

3. ***Maximizing Perimeter*** What is the smallest perimeter possible for a rectangle whose area is 16 in^2, and what are its dimensions? Smallest perimeter = 16 in., dimensions are 4 in. by 4 in.

4. ***Finding Area*** Show that among all rectangles with an 8-m perimeter, the one with largest area is a square. See page 232.

2



$h=5$

largest # $\rightarrow$ "A" $= \frac{1}{2} x y \leftarrow$ "primary"

$x^2+y^2=25 \leftarrow$ "secondary"

$y^2 = 25 - x^2$

$y = \sqrt{25-x^2}$

$f = \frac{1}{2}x \qquad g = (25-x^2)^{\frac{1}{2}}$

$f' = \frac{1}{2} \qquad g' = \frac{1}{2}(25-x^2)^{-\frac{1}{2}}(-2x)$

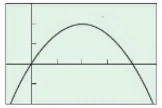
$\frac{d}{dx}\left[\frac{1}{2}x\left(\sqrt{25-x^2}\right)\right] \qquad \frac{d}{dx}\left[\#\right]$

$\left(\frac{1}{2}x\right)\left(\frac{1}{2}(25-x^2)^{-\frac{1}{2}}(-2x)\right) + \frac{1}{2}(25-x^2)^{\frac{1}{2}} = 0$

$x + y = 20$

$y = 20 - x$

$xy = \text{Max}$



[–5, 25] by [–100, 150]

Figure 4.35 The graph of $f(x) = x(20 - x)$ with domain $(-\infty, \infty)$ has an absolute maximum of 100 at $x = 10$. (Example 1)

EXAMPLE 1 Using the Strategy

Find two numbers whose sum is 20 and whose product is as large as possible.

SOLUTION

Model If one number is x, the other is $(20 - x)$, and their product is $f(x) = x(20 - x)$.

Solve Graphically We can see from the graph of f in Figure 4.35 that there is a maximum. From what we know about parabolas, the maximum occurs at $x = 10$.

Interpret The two numbers we seek are $x = 10$ and $20 - x = 10$.

Now try Exercise 1.

$20x - x^2 = \text{max}$

$20 - 2x = 0$

$+2x \quad +2x$

$20 = 2x$

$x = 10$

In Exercises 1–10, solve the problem analytically. Support your answer graphically.

$x + y = 20 \rightarrow y = 20 - x$

1. ***Finding Numbers*** The sum of two nonnegative numbers is 20. Find the numbers if

   (a) the sum of their squares is as large as possible; as small as possible.

   (b) one number plus the square root of the other is as large as possible; as small as possible.

a) $x^2 + y^2 = \text{max (or min)}$

$x^2 + (20-x)^2 = \text{max (or min)}$

$x^2 + 400 - 40x + x^2$

$2x^2 - 40x + 400 = \text{max (or min)}$

$4x - 40 = 0$

$x = 10 \;;\; y = 10$

$x = 0 \;;\; y = 20$

Think: most extreme situation...

$10^2 + 10^2 =$

$100 + 100 = 200$

$0^2 + 20^2$

$0 + 400 = 400$

b) $x + \sqrt{y} = \text{max (or min)}$

$x + \sqrt{20-x} = \text{max (or min)}$

$x + (20-x)^{\frac{1}{2}} = \text{max (or min)}$

$1 + \frac{1}{2}(20-x)^{-\frac{1}{2}}(-1) = 0$

$x \neq 20$

$1 - \frac{1}{2(20-x)^{\frac{1}{2}}} = 0$

$\frac{2(20-x)^{\frac{1}{2}}}{2(20-x)^{\frac{1}{2}}} - \frac{1}{2(20-x)^{\frac{1}{2}}} = \frac{2(20-x)^{\frac{1}{2}} - 1}{2(20-x)^{\frac{1}{2}}} = 0$

$2(20-x)^{\frac{1}{2}} - 1 = 0$

$\left((20-x)^{\frac{1}{2}}\right)^2 = \left(\frac{1}{2}\right)^2$

$\frac{1}{4} + \frac{3}{4} = \frac{3}{4}$

$20 - x = \frac{1}{4}$

$-x = \frac{1}{4} - \frac{80}{4}$

$x = \frac{79}{4}$

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③ $2x + 2y = \text{smallest}$

$xy = 16$

$y = \frac{16}{x}$

$2x + 2\left(\frac{16}{x}\right) = \text{smallest perimeter}$

$2x + \frac{32}{x} = \#$ $\quad 32x^{-1}$

$2 - 32x^{-2} = 0$

$2x^2 - 32 = 0 \qquad x = 4, -4$

$x^2 - 16 = 0$

$(x+4)(x-4) = 0$

$P = 8\text{m}$

$2x + 2y = 8$

$x + y = 4$

$xy = \text{max} = A$

$A = x(4-x)$

$y = 4 - x$