What you'll learn about

- Related Rate Equations
- Solution Strategy
- Simulating Related Motion

4.6 Related Rates

The general approach for this section is as follows;

1) create a formula related to the given problem,

2) derive, *with respect to time.* $\frac{d}{dt}\left(f(x)\right) = \frac{dx}{dt}$

implicit diff. !

3) Plug in (or find) given values (including rates!), solve for the indicated value.

...the first set of problems will practice the first two steps.

(a) $V = \frac{4}{3}\pi r^3$ Volume formula for a sphere

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

(b) $V = \frac{\pi}{3} r^2 h$ Cone volume formula

$$\frac{dV}{dt} = \frac{\pi}{3}\left(r^2 \cdot \frac{dh}{dt} + 2r\frac{dr}{dt} \cdot h\right) = \frac{\pi}{3}\left(r^2 \frac{dh}{dt} + 2rh\frac{dr}{dt}\right)$$

$\frac{d}{dt}$

$f = r^2$ $\quad g = h$

$f' = 2r \cdot \frac{dr}{dt}$ $\quad g' = \frac{dh}{dt}$

In Exercises 1–41, assume all variables are differentiable functions of t.

1. ***Area*** The radius r and area A of a circle are related by the equation $A = \pi r^2$. Write an equation that relates dA/dt to dr/dt. $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

2. ***Surface Area*** The radius r and surface area S of a sphere are related by the equation $S = 4\pi r^2$. Write an equation that relates dS/dt to dr/dt. $\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

3. ***Volume*** The radius r, height h, and volume V of a right circular cylinder are related by the equation $V = \pi r^2 h$.

   (a) How is dV/dt related to dh/dt if r is constant? $\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$

   (b) How is dV/dt related to dr/dt if h is constant? $\frac{dV}{dt} = 2\pi rh \frac{dr}{dt}$

   (c) How is dV/dt related to dr/dt and dh/dt if neither r nor h is constant? $\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} + 2\pi rh \frac{dr}{dt}$

4. ***Electrical Power*** The power P (watts) of an electric circuit is related to the circuit's resistance R (ohms) and current I (amperes) by the equation $P = RI^2$.

   (a) How is dP/dt related to dR/dt and dI/dt?

   (b) How is dR/dt related to dI/dt if P is constant?

1 $\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$

$1.\ \frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$

4 $\frac{d}{dt}P = \frac{d}{dt}(RI^2)$

a) $\frac{dP}{dt} = \frac{dR}{dt} I^2 + R \cdot 2I \frac{dI}{dt}$

b) $0 = \frac{dR}{dt} I^2 + R \cdot 2I \frac{dI}{dt}$

5. ***Diagonals*** If x, y, and z are lengths of the edges of a rectangular box, the common length of the box's diagonals is $s = \sqrt{x^2 + y^2 + z^2}$. How is ds/dt related to dx/dt, dy/dt, and dz/dt? See below.

6. ***Area*** If a and b are the lengths of two sides of a triangle, and θ the measure of the included angle, the area A of the triangle is $A = (1/2)\,ab\sin\theta$. How is dA/dt related to da/dt, db/dt, and $d\theta/dt$? $\frac{dA}{dt} = \frac{1}{2}\left(b\sin\theta\frac{da}{dt} + a\sin\theta\frac{db}{dt} + ab\cos\theta\frac{d\theta}{dt}\right)$

$$5.\ \frac{ds}{dt} = \frac{x\dfrac{dx}{dt} + y\dfrac{dy}{dt} + z\dfrac{dz}{dt}}{\sqrt{x^2 + y^2 + z^2}}$$

5

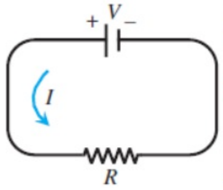
$$s = \sqrt{x^2 + y^2 + z^2}$$

$$s = (x^2 + y^2 + z^2)^{\frac{1}{2}}$$

$$\frac{ds}{dt} = \frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt}\right)$$

NOW, let's practice step 3; plug in values into the derived function!

7. ***Changing Voltage*** The voltage V (volts), current I (amperes), and resistance R (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 volt/sec while I is decreasing at the rate of $1/3$ amp/sec. Let t denote time in sec.



(a) 1 volt/sec

(b) $-\frac{1}{3}$ amp/sec

(c) $\frac{dV}{dt} = I\frac{dR}{dt} + R\frac{dI}{dt}$

(d) $\frac{dR}{dt} = \frac{3}{2}$ ohms/sec. R is increasing since $\frac{dR}{dt}$ is positive.

(a) What is the value of dV/dt?

(b) What is the value of dI/dt?

(c) Write an equation that relates dR/dt to dV/dt and dI/dt.

(d) **Writing to Learn** Find the rate at which R is changing when $V = 12$ volts and $I = 2$ amp. Is R increasing, or decreasing? Explain.

$$\frac{dv}{dt} = 1$$

$$\frac{dI}{dt} = \frac{-1}{3}$$

$$\frac{dR}{dt} = ?$$

$$V = 12$$
$$I = 2$$

$$12 = 2R$$
$$R = 6$$

c) $$\frac{dV}{dt} = I\frac{dR}{dt} + R\frac{dI}{dt}$$

$$1 = 2\left(\frac{dR}{dt}\right) + (6)\left(\frac{-1}{3}\right)$$

$$1 = 2\frac{dR}{dt} - 2$$

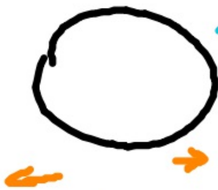
$$+2 \qquad +2$$

$$3 = 2\frac{dR}{dt}$$

$$\frac{dR}{dt} = \frac{3}{2}$$

8. ***Heating a Plate*** When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/sec. At what rate is the plate's area increasing when the radius is 50 cm? π cm^2/sec

$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$$



$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\frac{dr}{dt} = 0.01$$

$$r = 50$$

$$\frac{dA}{dt} = 2\pi(50)(0.01)$$

$$= \pi$$

$$\frac{dA}{dt}$$

9. *Changing Dimensions in a Rectangle* The length ℓ of a rectangle is decreasing at the rate of 2 cm/sec while the width w is increasing at the rate of 2 cm/sec. When $\ell = 12$ cm and $w = 5$ cm, find the rates of change of See page 255.

(a) the area, (b) the perimeter, and

(c) the length of a diagonal of the rectangle.

(d) **Writing to Learn** Which of these quantities are decreasing, and which are increasing? Explain.

$\frac{d\ell}{dt} = -2$

2×1, 2×0, 2^2

$\frac{dw}{dt} = 2$

a) $A = \ell w$

$\frac{dA}{dt} = \frac{d\ell}{dt} w + \frac{dw}{dt} \ell$

$f = \ell \qquad g = w$

$f' = \frac{d\ell}{dt} \qquad g' = \frac{dw}{dt}$

$\frac{dA}{dt} = (-2)(5) + (2)(12)$

$\frac{dA}{dt} = -10 + 24 = 14$

b) $\frac{d}{dt}(2w + 2\ell) = \frac{d}{dt}(P)$

$2\frac{dw}{dt} + 2\frac{d\ell}{dt} = \frac{dP}{dt}$

$2(2) + 2(-2) = \frac{dP}{dt}$

$0 = \frac{dP}{dt}$

9. *Changing Dimensions in a Rectangle* The length ℓ of a rectangle is decreasing at the rate of 2 cm/sec while the width w is increasing at the rate of 2 cm/sec. When $\ell = 12$ cm and $w = 5$ cm, find the rates of change of See page 255.

(a) the area, (b) the perimeter, and

(c) the length of a diagonal of the rectangle.

(d) **Writing to Learn** Which of these quantities are decreasing, and which are increasing? Explain.

10. *Changing Dimensions in a Rectangular Box* Suppose that the edge lengths x, y, and z of a closed rectangular box are changing at the following rates:

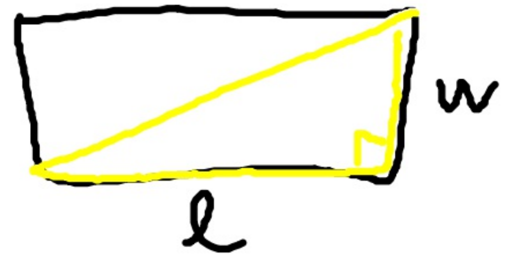
(a) 2 m^3/sec (b) 0 m^2/sec (c) 0 m/sec

$$\frac{dx}{dt} = 1 \text{ m/sec}, \quad \frac{dy}{dt} = -2 \text{ m/sec}, \quad \frac{dz}{dt} = 1 \text{ m/sec}.$$

Find the rates at which the box's (a) volume, (b) surface area, and (c) diagonal length $s = \sqrt{x^2 + y^2 + z^2}$ are changing at the instant when $x = 4$, $y = 3$, and $z = 2$.

$\frac{d\ell}{dt} = -2$ $\quad \frac{dw}{dt} = 2$

c)



a) $A = \ell w$

$$\frac{dA}{dt} = \frac{d\ell}{dt} w + \frac{dw}{dt} \ell$$

$$= (-2)(5) + (2)(12)$$

$$= -10 + 24$$

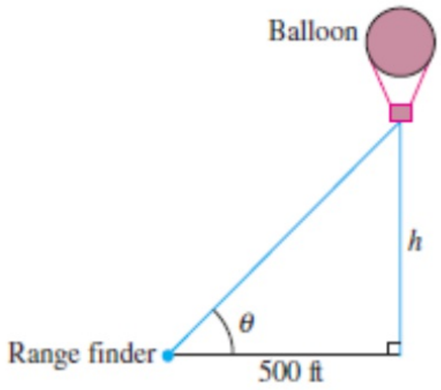
$$\frac{dA}{dt} = 14$$

b) $2\ell + 2w = P$

$$2\frac{d\ell}{dt} + 2\frac{dw}{dt} = \frac{dP}{dt}$$

EXAMPLE 2 A Rising Balloon

A hot-air balloon rising straight up from a level field is tracked by a range finder 500 feet from the lift-off point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 radians per minute. How fast is the balloon rising at that moment?



SOLUTION

We will carefully identify the six steps of the strategy in this first example.

Step 1: Let h be the height of the balloon and let θ be the elevation angle.

We seek: dh/dt

We know: $d\theta/dt = 0.14$ rad/min

Step 2: We draw a picture (Figure 4.55). We label the horizontal distance "500 ft" cause it does not change over time. We label the height "h" and the angle (evation "θ." Notice that we do not label the angle "$\pi/4$," as that would free the picture.

Step 3: We need a formula that relates h and θ. Since $\dfrac{h}{500} = \tan\theta$, we get $h = 500 \tan\theta$.

Step 4: Differentiate implicitly:

$$\frac{d}{dt}(h) = \frac{d}{dt}(500 \tan\theta)$$

$$\frac{dh}{dt} = 500 \sec^2\theta \, \frac{d\theta}{dt}$$

Step 5: Let $d\theta/dt = 0.14$ and let $\theta = \pi/4$. (Note that it is now safe to specify our moment in time.)

$$\frac{dh}{dt} = 500 \sec^2\left(\frac{\pi}{4}\right)(0.14) = 500(\sqrt{2})^2\,(0.14) = 140.$$

Step 6: At the moment in question, the balloon is rising at the rate of 140 ft/min.

Now try Exercise 11.

11. *Inflating Balloon* A spherical balloon is inflated with helium at the rate of 100π ft^3/min.

(a) How fast is the balloon's radius increasing at the instant the radius is 5 ft? 1 ft/min

(b) How fast is the surface area increasing at that instant? 40π ft^2/min

11) a) $\frac{dV}{dt} = 100\pi$

$V = \frac{4}{3}\pi r^3$

$\frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$

$100\pi = 4\pi(5)^2 \frac{dr}{dt}$

$\frac{100\pi}{100\pi} = \frac{100\pi}{100\pi}\frac{dr}{dt}$

$1 = \frac{dr}{dt}$

b) $S = 4\pi r^2$

$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

$r = 5$

$\frac{ds}{dt} = 8\pi(5)(1)$

$= 40\pi$

EXAMPLE 3 A Highway Chase

A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

SOLUTION

We carry out the steps of the strategy.

Let x be the distance of the speeding car from the intersection, let y be the distance of the police cruiser from the intersection, and let z be the distance between the car and the cruiser. Distances x and z are increasing, but distance y is decreasing; so dy/dt is negative.

We seek: dx/dt

We know: $dz/dt = 20$ mph and $dy/dt = -60$ mph

A sketch (Figure 4.56) shows that x, y, and z form three sides of a right triangle. We need to relate those three variables, so we use the Pythagorean Theorem:

$$x^2 + y^2 = z^2$$

Differentiating implicitly with respect to t, we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}, \text{ which reduces to } x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}.$$

We now substitute the numerical values for x, y, dz/dt, dy/dt, and z (which equals $\sqrt{x^2 + y^2}$):

$$(0.8)\frac{dx}{dt} + (0.6)(-60) = \sqrt{(0.8)^2 + (0.6)^2}(20)$$

$$(0.8)\frac{dx}{dt} - 36 = (1)(20)$$

$$\frac{dx}{dt} = 70$$

At the moment in question, the car's speed is 70 mph. *Now try Exercise 13.*

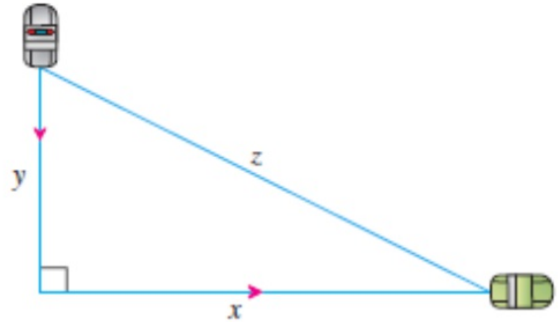
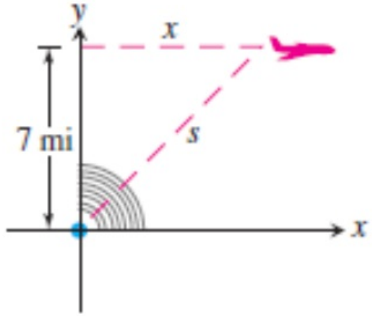


Figure 4.56 A sketch showing the variables in Example 3. We know dy/dt and dz/dt, and we seek dx/dt. The variables x, y, and z are related by the Pythagorean Theorem: $x^2 + y^2 = z^2$.

13. ***Air Traffic Control*** An airplane is flying at an altitude of 7 mi and passes directly over a radar antenna as shown in the figure. When the plane is 10 mi from the antenna ($s = 10$), the radar detects that the distance s is changing at the rate of 300 mph. What is the speed of the airplane at that moment? $\frac{dx}{dt} = \frac{3000}{\sqrt{51}}$ mph $\approx$ 420.08 mph



14. ***Flying a Kite*** Inge flies a kite at a height of 300 ft, the wind carrying the kite horizontally away at a rate of 25 ft/sec. How fast must she let out the string when the kite is 500 ft away from her? 20 ft/sec

EXAMPLE 4 Filling a Conical Tank

Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

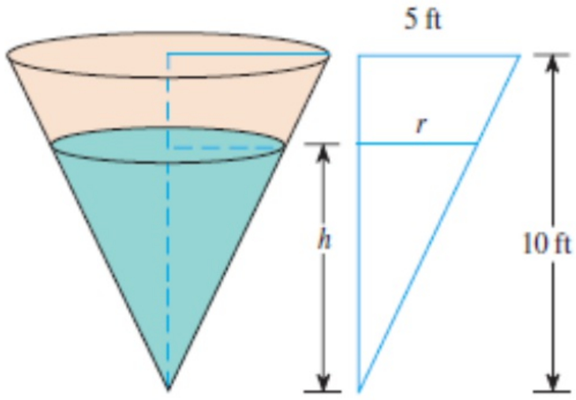


Figure 4.57 In Example 4, the cone of water is increasing in volume inside the reservoir. We know *dV/dt* and we seek *dh/dt*. Similar triangles enable us to relate *V* directly to *h*.

SOLUTION 1

We carry out the steps of the strategy. Figure 4.57 shows a partially filled conical tank. The tank itself does not change over time; what we are interested in is the changing cone of *water* inside the tank. Let V be the volume, r the radius, and h the height of the cone of water.

We seek: dh/dt

We know: $dV/dt = 9$ ft^3/min

We need to relate V and h. The volume of the cone of water is $V = \frac{1}{3}\pi r^2 h$, but this formula also involves the variable r, whose rate of change is not given. We need to either find dr/dt (see Solution 2) or eliminate r from the equation, which we can do by using the similar triangles in Figure 4.57 to relate r and h:

$$\frac{r}{h} = \frac{5}{10}, \text{ or simply } r = \frac{h}{2}.$$

Therefore,

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3.$$

Differentiate with respect to t:

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3h^2\frac{dh}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}.$$

Let $h = 6$ and $dV/dt = 9$; then solve for dh/dt:

$$9 = \frac{\pi}{4}(6)^2\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{\pi} \approx 0.32$$

At the moment in question, the water level is rising at 0.32 ft/min.

17. ***Draining Conical Reservoir*** Water is flowing at the rate of 50 m^3/min from a concrete conical reservoir (vertex down) of base radius 45 m and height 6 m. (a) How fast is the water level falling when the water is 5 m deep? (b) How fast is the radius of the water's surface changing at that moment? Give your answer in cm/min.

(a) $\frac{32}{9\pi} \approx 1.13$ cm/min (b) $-\frac{80}{3\pi} \approx -8.49$ cm/min

$h = 5$

$r = 45$

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$V = \frac{1}{3}\pi(\overset{f}{r^2}\overset{g}{h})$

a) $\frac{dh}{dt} = ?$ $\frac{dV}{dt} = 50$

$\frac{h}{r} = \frac{6}{45}$

$f = r^2 \quad g = h$

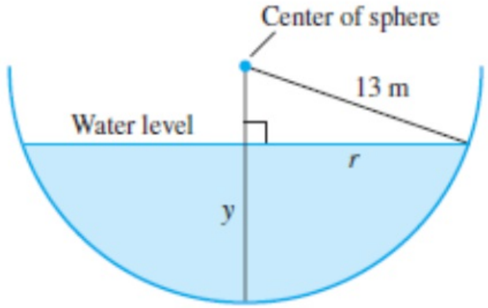
$f' = 2r \quad g' = \frac{dh}{dt}$

$r = 45 \quad h = 6$

$\frac{dV}{dt} = \frac{1}{3}\pi\left(2r\frac{dr}{dt}h + \frac{dh}{dt}r^2\right)$

$50 = \frac{1}{3}\pi\left(2(45)(6)\frac{dr}{dt} + \frac{dh}{dt}(45)^2\right)$

18. *Draining Hemispherical Reservoir* Water is flowing at the rate of 6 m^3/min from a reservoir shaped like a hemispherical bowl of radius 13 m, shown here in profile. Answer the following questions given that the volume of water in a hemispherical bowl of radius R is $V = (\pi/3)y^2(3R - y)$ when the water is y units deep.



(a) At what rate is the water level changing when the water is 8 m deep? $-\dfrac{1}{24\pi} \approx -0.01326$ m/min or $-\dfrac{25}{6\pi} \approx -1.326$ cm/min

(b) What is the radius r of the water's surface when the water is y m deep? $r = \sqrt{169 - (13 - y)^2} = \sqrt{26y - y^2}$

(c) At what rate is the radius r changing when the water is 8 m deep?
$-\dfrac{5}{288\pi} \approx -0.00553$ m/min or $-\dfrac{125}{72\pi} \approx -0.553$ cm/min

19. *Sliding Ladder* A 13-ft ladder is leaning against a house (see figure) when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.

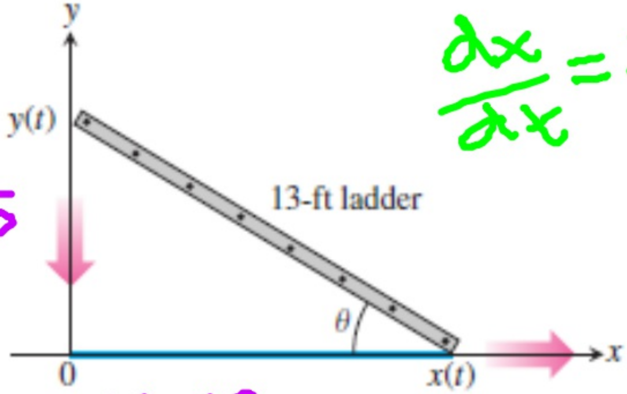
$\frac{dx}{dt} = 5$

$x^2 + y^2 = r^2$

$x^2 + y^2 = 13^2$

$x^2 + y^2 = 169$

$y=5$



$x=12$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$2(12)(5) + 2(5)\frac{dy}{dt} = 0$

$120 + 10\frac{dy}{dt} = -120$

$-120 \quad -120$

$\frac{10}{10}\frac{dy}{dt} = \frac{-120}{10}$

$\frac{dy}{dt} = -12$

(a) How fast is the top of the ladder sliding down the wall at that moment? 12 ft/sec

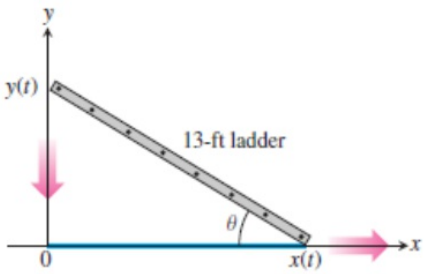
(b) At what rate is the area of the triangle formed by the ladder, wall, and ground changing at that moment? $-\frac{119}{2}$ ft²/sec

(c) At what rate is the angle θ between the ladder and the ground changing at that moment? -1 radian/sec

b) $A = \frac{1}{2}xy$

c) $\cos\theta = \frac{x}{13}$

19. ***Sliding Ladder*** A 13-ft ladder is leaning against a house (see figure) when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.



(a) How fast is the top of the ladder sliding down the wall at that moment? 12 ft/sec

(b) At what rate is the area of the triangle formed by the ladder, wall, and ground changing at that moment? $-\frac{119}{2}$ ft²/sec

(c) At what rate is the angle θ between the ladder and the ground changing at that moment? -1 radian/sec