1 Identify Rational Zeros Usually it is not practical to test all possible zeros of a polynomial function using synthetic substitution. The **Rational Zero Theorem** can help you choose some possible zeros to test. If the leading coefficient is 1, the corollary applies.

5-8 Rational Roots Theorem

KeyConcept Rational Zero Theorem

Words	If $P(x)$ is a polynomial function with integral coefficients, then every rational zero of $P(x) = 0$ is of the form $\frac{p}{q}$, a rational number in simplest form, where p is a factor of the constant term and q is a factor of the leading coefficient.
Example	Let $f(x) = 6x^4 + 22x^3 + 11x^2 - 80x - 40$. If $\frac{4}{3}$ is a zero of $f(x)$, then 4 is a factor of -40, and 3 is a factor of 6.

Corollary to the Rational Zero Theorem

If $P(x)$ is a polynomial function with integral coefficients, a leading coefficient of 1, and a nonzero constant term, then any rational zeros of $P(x)$ must be factors of the constant term.

$2x + 3 = 0$ $\quad x = \frac{-3}{2}$

$-3 \quad -3$

$2x = -3$

$4x - 5 = 0$ $\quad x = \frac{5}{4}$

$+5 \quad +5$

$4x = 5$

$(2x+3)(4x-5) = 0$

$8x^2 - 10x + 12x - 15$

$8x^2 + 2x - 15 = 0$

Factors

Example 1 Identify Possible Zeros

List all of the possible rational zeros of each function.

a. $f(x) = 4x^5 + x^4 - 2x^3 - 5x^2 + 8x + 16$

If $\frac{p}{q}$ is a rational zero, then p is a factor of 16 and q is a factor of 4.

p: ±1, ±2, ±4, ±8, ±16 $\quad$ q: ±1, ±2, ±4

Write the possible values of $\frac{p}{q}$ in simplest form.

$\frac{p}{q} = \pm1, \pm2, \pm4, \pm8, \pm16, \pm\frac{1}{2}, \pm\frac{1}{4}$

b. $f(x) = x^3 - 2x^2 + 5x + 12$

If $\frac{p}{q}$ is a rational zero, then p is a factor of 12 and q is a factor of 1.

p: ±1, ±2, ±3, ±4, ±6, ±12 $\quad$ q: ±1

So, $\frac{p}{q} = \pm1, \pm2, \pm3, \pm4, \pm6$, and ±12

$\frac{p}{q}$

Check Your Understanding

= Step-by-Step Solutions begin on page R14.

Example 1 **List all of the possible rational zeros of each function.**

1. $f(x) = x^3 - 6x^2 - 8x + 24$

±1, ±2, ±3, ±4, ±6, ±8, ±12, ±24

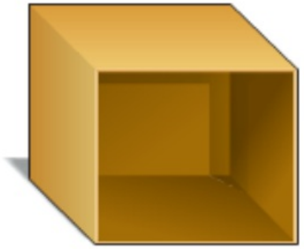
2. $f(x) = 2x^4 + 3x^2 - x + 15$

$p = \pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24$

$q = \pm1$

Real-World Example 2 Find Rational Zeros

WOODWORKING **Adam is building a computer desk with a separate compartment for the computer. The compartment for the computer is a rectangular prism and will be 8019 cubic inches. The compartment will be 24 inches longer than it is wide and the height will be 18 inches greater than the width. Find the dimensions of the computer compartment.**



Let x = width, $x + 24$ = length, and $x + 18$ = height.

Write an equation for the volume.

$\ell wh = V$	Formula for volume
$(x + 24)(x)(x + 18) = 8019$	Substitute.
$x^3 + 42x^2 + 432x = 8019$	Multiply.
$x^3 + 42x^2 + 432x - 8019 = 0$	Subtract 8019 from each side.

The leading coefficient is 1, so the possible rational zeros are factors of 8019.

±1, ±3, ±9, ±11, ±27, ±33, ±81, ±99, ±243, ±297, ±729, ±891, ±2673, and ±8019

Since length can only be positive, we only need to check positive values.

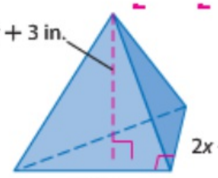
There is one change of sign of the coefficients, so by Descartes' Rule of Signs, there is only one positive real zero. Make a table for synthetic division and test possible values.

p	1	42	432	−8019
1	1	43	475	−7544
2	1	45	567	−6318
9	1	51	891	0

One zero is 9. Since there is only one positive real zero, we do not have to test the other numbers. The other dimensions are 9 + 24 or 33 inches, and 9 + 18 or 27 inches.

Example 2

3. **CCSS REASONING** The volume of the triangular pyramid is 210 cubic inches. Find the dimensions of the solid.
5 in. × 9 in. × 28 in.



q

p

(5) $2x^4+11x^3+26x^2+29x+12$

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

$\frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm \frac{1}{2}, \pm \frac{3}{2}$

$\pm 1, \pm 2$

Example 3 Find All Zeros

Find all of the zeros of $f(x) = 5x^4 - 8x^3 + 41x^2 - 72x - 36$.

From the corollary to the Fundamental Theorem of Algebra, there are exactly 4 complex zeros. According to Descartes' Rule of Signs, there are 3 or 1 positive real zeros and exactly 1 negative real zero. The possible rational zeros are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36, \pm \frac{1}{5}, \pm \frac{2}{5}, \pm \frac{3}{5}, \pm \frac{4}{5}, \pm \frac{6}{5}, \pm \frac{9}{5}, \pm \frac{12}{5}, \pm \frac{18}{5}$, and $\pm \frac{36}{5}$.

Make a table and test some possible rational zeros.

$\frac{p}{q}$	5	−8	41	−72	−36
−1	5	−13	54	−126	90
1	5	−3	38	−34	−70
2	5	2	45	18	0

Because $f(2) = 0$, there is a zero at $x = 2$. Factor the depressed polynomial $5x^3 + 2x^2 + 45x + 18$.

$5x^3 + 2x^2 + 45x + 18 = 0$ Write the depressed polynomial.

$(5x^3 + 2x^2) + (45x + 18) = 0$ Group terms.

$x^2(5x + 2) + 9(5x + 2) = 0$ Factor.

$(x^2 + 9)(5x + 2) = 0$ Distributive Property

$x^2 + 9 = 0$ or $5x + 2 = 0$ Zero Product Property

$x^2 = -9$ $\quad$ $5x = -2$

$x = \pm 3i$ $\quad$ $x = -\frac{2}{5}$

There is another real zero at $x = -\frac{2}{5}$ and two imaginary zeros at $x = 3i$ and $x = -3i$.

The zeros of the function are $-\frac{2}{5}$, 2, $3i$, and $-3i$.

Example 3 Find all of the zeros of each function.

6. $f(x) = 3x^3 - 2x^2 - 8x + 5$

7. $f(x) = 8x^3 + 14x^2 + 11x + 3$

8. $f(x) = 4x^4 + 13x^3 - 8x^2 + 13x - 12$

9. $f(x) = 4x^4 - 12x^3 + 25x^2 - 14x - 15$

Example 1 List all of the possible rational zeros of each function. **10–17. See margin.**

10. $f(x) = x^4 + 8x - 32$

11. $f(x) = x^3 + x^2 - x - 56$

12. $f(x) = 2x^3 + 5x^2 - 8x - 10$

13. $f(x) = 3x^6 - 4x^4 - x^2 - 35$

14. $f(x) = 6x^5 - x^4 + 2x^3 - 3x^2 + 2x - 18$

15. $f(x) = 8x^4 - 4x^3 - 4x^2 + x + 42$

16. $f(x) = 15x^3 + 6x^2 + x + 90$

17. $f(x) = 16x^4 - 5x^2 + 128$

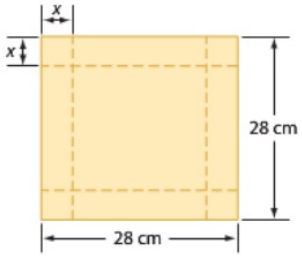
Example 2

18. **MANUFACTURING** A box is to be constructed by cutting out equal squares from the corners of a square piece of cardboard and turning up the sides. **a. See margin.**

 a. Write a function $V(x)$ for the volume of the box.

 b. For what value of x will the volume of the box equal 1152 cubic centimeters? **2 or 8**

 c. What will be the volume of the box if $x = 6$ centimeters? **1536 cm^3**



Find all of the rational zeros of each function.

19. $f(x) = x^3 + 10x^2 + 31x + 30$ **$-5, -3, -2$**

20. $f(x) = x^3 - 2x^2 - 56x + 192$ **$-8, 4, 6$**

21. $f(x) = 4x^3 - 3x^2 - 100x + 75$ **$-5, \frac{3}{4}, 5$**

22. $f(x) = 4x^4 + 12x^3 - 5x^2 - 21x + 10$ **$-\frac{5}{2}, -2, \frac{1}{2}, 1$**

23. $f(x) = x^4 + x^3 - 8x - 8$ **$-1, 2$**

24. $f(x) = 2x^4 - 3x^3 - 24x^2 + 4x + 48$ **$-2, 4, \frac{3}{2}$**

25. $f(x) = 4x^3 + x^2 + 16x + 4$ **$-\frac{1}{4}$**

26. $f(x) = 81x^4 - 256$ **$-\frac{4}{3}, \frac{4}{3}$**

Additional Answers

10. $\pm1, \pm2, \pm4, \pm8, \pm16, \pm32$

11. $\pm1, \pm2, \pm4, \pm7, \pm8, \pm14, \pm28, \pm56$

12. $\pm1, \pm2, \pm5, \pm10, \pm\frac{1}{2}, \pm\frac{5}{2}$

13. $\pm1, \pm5, \pm7, \pm35, \pm\frac{1}{3}, \pm\frac{5}{3}, \pm\frac{7}{3}, \pm\frac{35}{3}$

14. $\pm1, \pm2, \pm3, \pm6, \pm9, \pm18, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{9}{2}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{1}{6}$

15. $\pm1, \pm2, \pm3, \pm6, \pm7, \pm14, \pm21, \pm42, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{7}{2}, \pm\frac{21}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{7}{4}, \pm\frac{21}{4}, \pm\frac{1}{8}, \pm\frac{3}{8}, \pm\frac{7}{8}, \pm\frac{21}{8}$

16. $\pm1, \pm2, \pm3, \pm5, \pm6, \pm9, \pm10, \pm15, \pm18, \pm30, \pm45, \pm90, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{5}{3}, \pm\frac{10}{3}, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{6}{5}, \pm\frac{9}{5}, \pm\frac{18}{5}, \pm\frac{1}{15}, \pm\frac{2}{15}$

17. $\pm1, \pm2, \pm4, \pm8, \pm16, \pm32, \pm64, \pm128, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{8}, \pm\frac{1}{16}$

18a. $V(x) = (28 - 2x)(28 - 2x)x = 4x^3 - 112x^2 + 784x$

Example 3 **Find all of the zeros of each function.**

27. $f(x) = x^3 + 3x^2 - 25x + 21$ **−7, 1, 3**

28. $f(x) = 6x^3 + 5x^2 - 9x + 2$ $\frac{2}{3}, \frac{-3 \pm \sqrt{17}}{4}$

29. $f(x) = x^4 - x^3 - x^2 - x - 2$ **2, −1, *i*, −*i***

30. $f(x) = 10x^3 - 17x^2 - 7x + 2$

31. $f(x) = x^4 - 3x^3 + x^2 - 3x$ **0, 3, −*i*, *i***

32. $f(x) = 6x^3 + 11x^2 - 3x - 2$ $\frac{1}{2}, -\frac{1}{3}, -2$

33. $f(x) = 6x^4 + 22x^3 + 11x^2 - 38x - 40$

34. $f(x) = 2x^3 - 7x^2 - 8x + 28$ $-2, 2, \frac{7}{2}$

35. $f(x) = 9x^5 - 94x^3 + 27x^2 + 40x - 12$

36. $f(x) = x^5 - 2x^4 - 12x^3 - 12x^2 - 13x - 10$

37. $f(x) = 48x^4 - 52x^3 + 13x - 3$

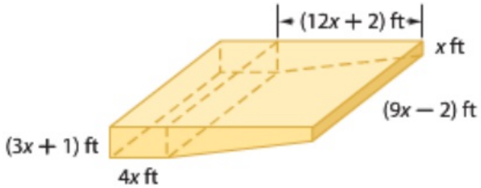
38. $f(x) = 5x^4 - 29x^3 + 55x^2 - 28x$ $\frac{4}{5}, 0, \frac{5 \pm i\sqrt{3}}{2}$

30. $-\frac{1}{2}, \frac{1}{5}, 2$

33. $-2, \frac{4}{3}, \frac{-3 \pm i}{2}$ **35.** $3, \frac{2}{3}, -\frac{2}{3}, \frac{-3 \pm \sqrt{13}}{2}$ **36.** **−1, −2, 5, *i*, −*i***

39. SWIMMING POOLS A diagram of the swimming pool at the Midtown Community Center is shown below. The pool can hold 9175 cubic feet of water.

37. $-\frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \frac{3}{4}$



a. $V(x) = 324x^3 + 54x^2 - 19x - 2$

b. $\frac{-57 \pm i\sqrt{8987}}{36}$, 3; 3 is the only reasonable value for x. The other two values are imaginary.