


5.3 Definite Integrals and Antiderivatives



Table 5.3 Rules for Definite Integrals

1. *Order of Integration:* $\int_b^a f(x) dx = -\int_a^b f(x) dx$ A definition 

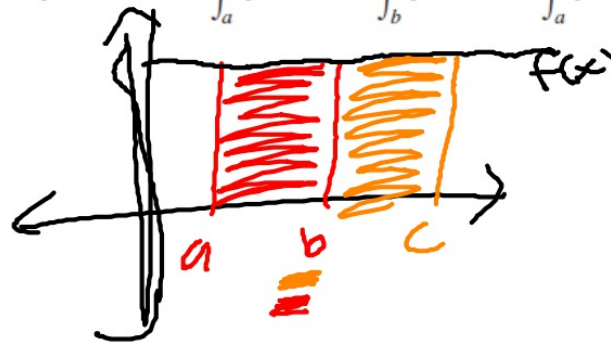
2. *Zero:* $\int_a^a f(x) dx = 0$ Also a definition

3. *Constant Multiple:* $\int_a^b kf(x) dx = k \int_a^b f(x) dx$ Any number k

$$\int_a^b -f(x) dx = -\int_a^b f(x) dx \quad k = -1$$

4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$

5. *Additivity:* $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$



EXAMPLE 1 Using the Rules for Definite Integrals

Suppose

$$\int_{-1}^1 f(x) dx = 5, \quad \int_1^4 f(x) dx = -2, \quad \text{and} \quad \int_{-1}^1 h(x) dx = 7.$$

Find each of the following integrals, if possible.

(a) $\int_4^1 f(x) dx$ (b) $\int_{-1}^4 f(x) dx$ (c) $\int_{-1}^1 [2f(x) + 3h(x)] dx$
(d) $\int_0^1 f(x) dx$ (e) $\int_{-2}^2 h(x) dx$ (f) $\int_{-1}^4 [f(x) + h(x)] dx$

SOLUTION

$$(a) \int_4^1 f(x) dx = -\int_1^4 f(x) dx = -(-2) = 2$$

$$(b) \int_{-1}^4 f(x) dx = \int_{-1}^1 f(x) dx + \int_1^4 f(x) dx = 5 + (-2) = 3$$

$$(c) \int_{-1}^1 [2f(x) + 3h(x)] dx = 2 \int_{-1}^1 f(x) dx + 3 \int_{-1}^1 h(x) dx = 2(5) + 3(7) = 31$$

(d) Not enough information given. (We cannot assume, for example, that integrating over half the interval would give half the integral!)

(e) Not enough information given. (We have no information about the function h outside the interval $[-1, 1]$.)

(f) Not enough information given (same reason as in part (e)). *Now try Exercise 1.*

The exercises in this section are designed to reinforce your understanding of the definite integral from the algebraic and geometric points of view. For this reason, you should not use the numerical integration capability of your calculator (NINT) except perhaps to support an answer.

1. Suppose that f and g are continuous functions and that

$$\int_1^2 f(x) dx = -4, \quad \int_1^5 f(x) dx = 6, \quad \int_1^5 g(x) dx = 8.$$

Use the rules in Table 5.3 to find each integral.

$$\begin{aligned} \text{(a)} \int_2^2 g(x) dx &= 0 & \text{(b)} \int_5^1 g(x) dx &= -8 \\ \text{(c)} \int_1^2 3f(x) dx &= -12 & \text{(d)} \int_2^5 f(x) dx &= 10 \\ \text{(e)} \int_1^5 [f(x) - g(x)] dx &= -2 & \text{(f)} \int_1^5 [4f(x) - g(x)] dx &= 16 \end{aligned}$$

2. Suppose that f and h are continuous functions and that

$$\int_1^9 f(x) dx = -1, \quad \int_7^9 f(x) dx = 5, \quad \int_7^9 h(x) dx = 4.$$

Use the rules in Table 5.3 to find each integral.

$$\begin{aligned} \text{(a)} \int_1^9 -2f(x) dx &= 2 & \text{(b)} \int_7^9 [f(x) + h(x)] dx &= 9 \\ \text{(c)} \int_7^9 [2f(x) - 3h(x)] dx &= -2 & \text{(d)} \int_9^1 f(x) dx &= 1 \\ \text{(e)} \int_1^7 f(x) dx &= -6 & \text{(f)} \int_9^7 [h(x) - f(x)] dx &= 1 \end{aligned}$$

3. Suppose that $\int_1^2 f(x) dx = 5$. Find each integral.

$$\begin{aligned} \text{(a)} \int_1^2 f(u) du &= 5 & \text{(b)} \int_1^2 \sqrt{3} f(z) dz &= 5\sqrt{3} \\ \text{(c)} \int_2^1 f(t) dt &= -5 & \text{(d)} \int_1^2 [-f(x)] dx &= -5 \end{aligned}$$

4. Suppose that $\int_{-3}^0 g(t) dt = \sqrt{2}$. Find each integral.

$$\begin{aligned} \text{(a)} \int_0^{-3} g(t) dt &= -\sqrt{2} & \text{(b)} \int_{-3}^0 g(u) du &= \sqrt{2} \\ \text{(c)} \int_{-3}^0 [-g(x)] dx &= \sqrt{2} & \text{(d)} \int_{-3}^0 \frac{g(r)}{\sqrt{2}} dr &= 1 \end{aligned}$$

5. Suppose that f is continuous and that

$$\int_0^3 f(z) dz = 3 \quad \text{and} \quad \int_0^4 f(z) dz = 7.$$

Find each integral.

$$\text{(a)} \int_3^4 f(z) dz = 4 \quad \text{(b)} \int_4^3 f(t) dt = -4$$

6. Suppose that h is continuous and that

$$\int_{-1}^1 h(r) dr = 0 \quad \text{and} \quad \int_{-1}^3 h(r) dr = 6.$$

Find each integral.

$$\text{(a)} \int_1^3 h(r) dr = 6 \quad \text{(b)} -\int_3^1 h(u) du = 6$$

skip #7-14!

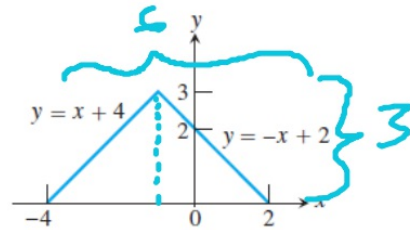
DEFINITION Average (Mean) Value

If f is integrable on $[a, b]$, its average (mean) value on $[a, b]$ is

$$av(f) = \frac{1}{b-a} \int_a^b f(x) dx.$$

In Exercises 15–18, find the average value of the function on the interval without integrating, by appealing to the geometry of the region between the graph and the x -axis.

15. $f(x) = \begin{cases} x+4, & -4 \leq x \leq -1, \\ -x+2, & -1 < x \leq 2, \end{cases}$ on $[-4, 2]$ $\frac{3}{2}$



Handwritten calculations for problem 15:

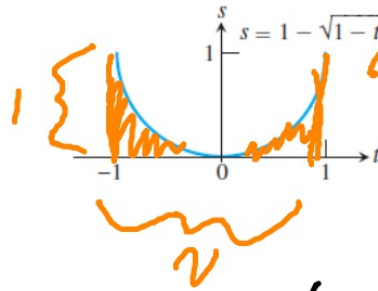
$$A = \frac{1}{2}(6)(3)$$

$$A = 9 = \int_a^b f(x) dx$$

$$\frac{1}{6} = \frac{1}{b-a}$$

$$av(f) = \frac{9}{6} = \frac{3}{2}$$

16. $f(t) = 1 - \sqrt{1-t^2}$, $[-1, 1]$ $\frac{4-\pi}{4}$



Handwritten calculations for problem 16:

$$av(f) = \frac{1}{2} \left(2 - \frac{\sqrt{3}}{2} \right)$$

$$= 1 - \frac{\sqrt{3}}{4}$$

Handwritten calculations for problem 16:

$$2(1) - \frac{\sqrt{3}}{2} = 2 - \frac{\sqrt{3}}{2}$$

$$\frac{1}{2} = \frac{1}{b-a}$$

$$av(f) = \frac{2 - \frac{\sqrt{3}}{2}}{2} = \frac{4 - \sqrt{3}}{4}$$

17. $f(t) = \sin t$, $[0, 2\pi]$ 0

18. $f(\theta) = \tan \theta$, $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ 0

