5.4 The Fundamental Theorem of Calculus

THEOREM 4 The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a, b]$, then the function

$$F(x) = \int_a^x f(t)\,dt$$

has a derivative at every point x in $[a, b]$, and

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

This says two things;

1) ALL functions have an antiderivative.

2) derivatives and integration are inverses to each other.

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x). \qquad (1)$$

EXAMPLE 1 Applying the Fundamental Theorem

Find

$$\frac{d}{dx}\int_{-\pi}^{x} \cos t\,dt \quad \text{and} \quad \frac{d}{dx}\int_0^x \frac{1}{1+t^2}\,dt$$

by using the Fundamental Theorem.

SOLUTION

$$\frac{d}{dx}\int_{-\pi}^{x} \cos t\,dt = \cos x \qquad \text{Eq. 1 with } f(t) = \cos t$$

$$\frac{d}{dx}\int_0^x \frac{1}{1+t^2}\,dt = \frac{1}{1+x^2}. \qquad \text{Eq. 1 with } f(t) = \frac{1}{1+t^2}$$

Now try Exercise 3.

WAIT...doesn't it matter where the integration starts for these...?

Apply the definition of the derivative directly to the function F. That is,

$$\frac{dF}{dx} = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt}{h}$$

$$= \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$

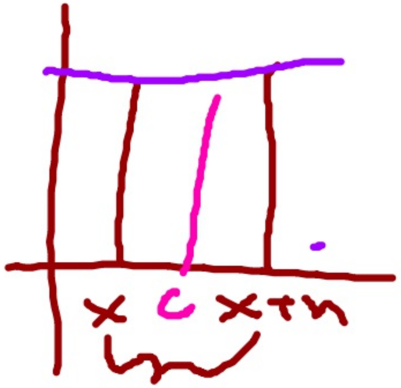
$$= \lim_{h\to 0} \left[\frac{1}{h}\int_x^{x+h} f(t)\,dt\right].$$

$$F(x) = \int_a^x f(t)\,dt$$

This must be constant!

$$\int_{a}^{x+h} f(t)\,dt + \int_x^{a} f(x)\,dt$$

A: nope.



h approaches zero!!

Rules for integrals, Section 5.3

$$\frac{dF}{dx} = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

$$= \lim_{h\to 0} f(c), \qquad \text{where } c \text{ lies between } x \text{ and } x+h.$$

What happens to c as h goes to zero? As $x+h$ gets closer to x, it carries c along with it like a bead on a wire, forcing c to approach x. Since f is continuous, this means that $f(c)$ approaches $f(x)$:

$$\lim_{h\to 0} f(c) = f(x).$$

BTW, "a" is a constant. BUT what if it's not...?

Also, what if there's some weird stuff going on in the limits of intergration...?

In Exercises 1–20, find dy/dx.

1. $y = \int_0^x (\sin^2 t)\, dt$ $\quad \sin^2 x$

constant!!

2. $y = \int_2^x (3t + \cos t^2)\, dt$

3. $y = \int_0^x (t^3 - t)^5\, dt$

4. $y = \int_{-2}^x \sqrt{1 + e^{5t}}\, dt$

5. $y = \int_2^x (\tan^3 u) du$

6. $y = \int_4^x e^u \sec u\, du$ $\quad e^x \sec x$

7. $y = \int_7^x \frac{1 + t}{1 + t^2} dt$

8. $y = \int_{-\pi}^x \frac{2 - \sin t}{3 + \cos t}\, dt$ $\quad \frac{2 - \sin x}{3 + \cos x}$

EXAMPLE 2 The Fundamental Theorem with the Chain Rule

Find dy/dx if $y = \int_1^{x^2} \cos t\, dt$.

$$\frac{d}{dx} f(g(x))$$

SOLUTION

The upper limit of integration is not x but x^2. This makes y a composite of

$$y = \int_1^u \cos t\, dt \quad \text{and} \quad u = x^2.$$

$$= f'(g(x)) \cdot g'(x)$$

We must therefore apply the Chain Rule when finding dy/dx.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \left(\frac{d}{du} \int_1^u \cos t\, dt \right) \cdot \frac{du}{dx}$$

$$= \cos u \cdot \frac{du}{dx}$$

$$= \cos(x^2) \cdot 2x$$

$$= 2x \cos x^2$$

Now try Exercise 9.

find $\frac{dy}{dx}$

9. $y = \int_0^{x^2} e^{t^2}\,dt$ $\boxed{}$

10. $y = \int_6^{x^2} \cot 3t\,dt$ $2x \cot 3x^2$

$u = x^2$

$du = 2x\,dx$

11. $y = \int_2^{5x} \frac{\sqrt{1+u^2}}{u}\,du$ $\boxed{}$

12. $y = \int_\pi^{\pi - x} \frac{1+\sin^2 u}{1+\cos^2 u}\,du$ $\leftarrow g(x)$

$u = \pi - x$

$du = -1\,dx$

13. $y = \int_x^6 \ln(1+t^2)\,dt$ $-\ln(1+x^2)$

14. $y = \int_x^7 \sqrt{2t^4+t+1}\,dt$ $\boxed{}$

12. $-\frac{1+\sin^2(\pi - x)}{1+\cos^2(\pi - x)}$

$-\int_6^x \ln(1+t^2)\,dt$

$\frac{d}{dx} f(g(x))$

$= f'(g(x))\, g'(x)$

$f'(u) \cdot u'$