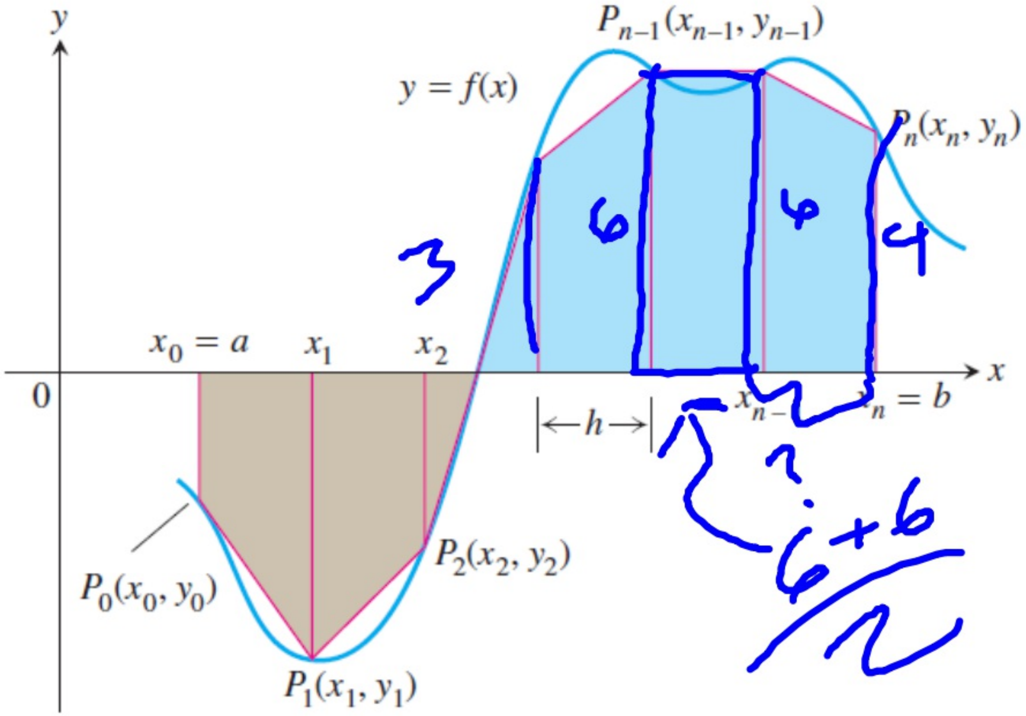
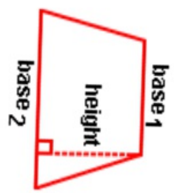
5.5 Trapazoidal Rule

We could get a better approximation is we were to use trapazoids instead of rectangles.

$$A=\frac{(b_1+b_2)\cdot h}{2} \quad \text{or} \quad A=\frac{1}{2}\cdot(b_1+b_2)\cdot h$$





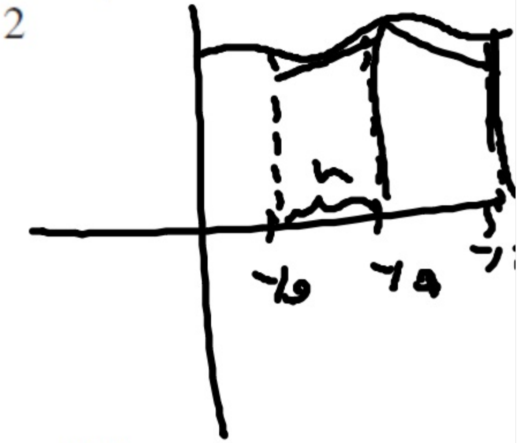
...btw, remember riemann's sum?

The region between the curve and the x-axis is then approximated by the trapezoids, the area of each trapezoid being the length of its horizontal "altitude" times the average of its two vertical "bases." That is,

$$\int_a^b f(x)\,dx \approx h \cdot \frac{y_0 + y_1}{2} + h \cdot \frac{y_1 + y_2}{2} + \cdots + h \cdot \frac{y_{n-1} + y_n}{2}$$

$$= h\left(\frac{y_0}{2} + y_1 + y_2 + \cdots + y_{n-1} + \frac{y_n}{2}\right)$$

$$= \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right),$$



where

$$y_0 = f(a), \quad y_1 = f(x_1), \quad \cdots, \quad y_{n-1} = f(x_{n-1}), \quad y_n = f(b).$$

This is algebraically equivalent to finding the numerical average of LRAM and RRAM; indeed, that is how some texts define the Trapezoidal Rule.

The Trapezoidal Rule

To approximate $\int_a^b f(x)\,dx$, use

$$T = \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right),$$

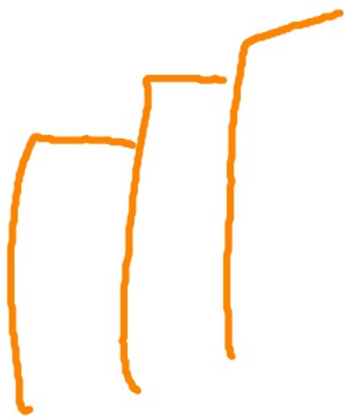
where $[a, b]$ is partitioned into n subintervals of equal length $h = (b - a)/n$.

Equivalently,

"left-side" "right-side"

$$T = \frac{\boxed{\mathrm{LRAM}_n + \mathrm{RRAM}_n}}{2},$$

where LRAM_n and RRAM_n are the Riemann sums using the left and right endpoints, respectively, for f for the partition.



EXAMPLE 1 Applying the Trapezoidal Rule

Use the Trapezoidal Rule with $n = 4$ to estimate $\int_1^2 x^2\,dx$. Compare the estimate with the value of NINT $(x^2, x, 1, 2)$ and with the exact value.

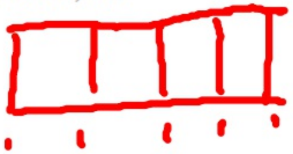
Table 5.4

x	$y = x^2$	
1	1	← y_0
$\frac{5}{4}$	$\frac{25}{16}$	← y_1
$\frac{6}{4}$	$\frac{36}{16}$	← y_2
$\frac{7}{4}$	$\frac{49}{16}$	← y_3
2	4	← y_4

SOLUTION

Partition $[1, 2]$ into four subintervals of equal length (Figure 5.32). Then evaluate $y = x^2$ at each partition point (Table 5.4).

Using these y values, $n = 4$, and $h = (2 - 1)/4 = 1/4$ in the Trapezoidal Rule, we have



$$T = \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right)$$

$$= \frac{1}{8}\left(1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4\right)$$

$$= \frac{75}{32} = 2.34375.$$

The value of $\boxed{\text{NINT}\ (x^2, x, 1, 2)}$ is 2.333333333.

The exact value of the integral is

$$\int_1^2 x^2\,dx = \frac{x^3}{3}\Big]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

The T approximation overestimates the integral by about half a percent of its true value of 7/3. The percentage error is $(2.34375 - 7/3)/(7/3) \approx 0.446\%$. *Now try Exercise 3.*

lower bound

upper bound

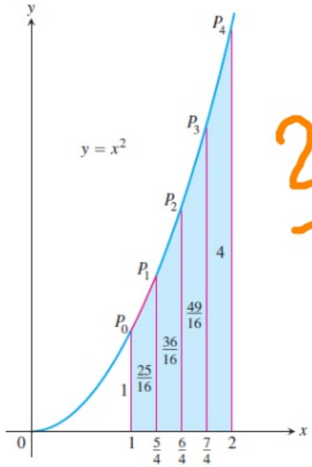
$\Delta x = \dfrac{b-a}{n}$

NINT$(x^2, x, 1, 2)$

$f(x)$ (integrand)

variable

$\dfrac{2-1}{4} = \dfrac{1}{4} = h$



In Exercises 1–6, (a) use the Trapezoidal Rule with $n = 4$ to approximate the value of the integral. (b) Use the concavity of the function to predict whether the approximation is an overestimate or an underestimate. Finally, (c) find the integral's exact value to check your answer.

1. $\int_0^2 x\,dx$ (a) 2 (b) Exact (c) 2

3. $\int_0^2 x^3\,dx$ (a) 4.25 (b) Over (c) 4

5. $\int_0^4 \sqrt{x}\,dx$ (a) 5.146 (b) Under (c) $\frac{16}{3}$

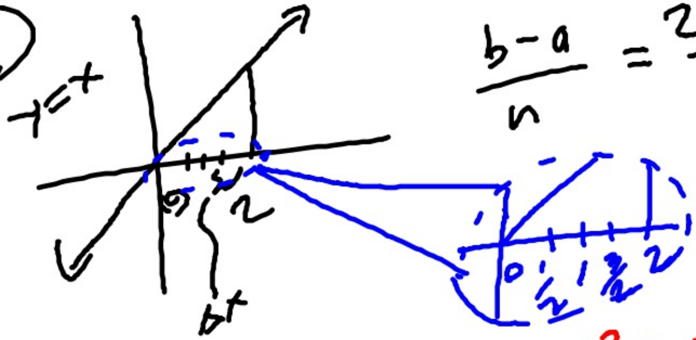
$y = x$

$y_0 = 0$
$y_1 = 1/2$
$y_2 = 1$
$y_3 = 3/2$
$y_4 = 2$

c) $\int_0^2 x\,dx = \left.\frac{1}{2}x^2\right|_0^2 = \frac{1}{2}(2)^2 - \frac{1}{2}(0)^2$

$F(b) - F(a)$

1



$\frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2} = \Delta x = h$

$(n=4)$

$\frac{h}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4)$

$\frac{\frac{1}{2}}{2}\left((0) + 2\left(\frac{1}{2}\right) + 2(1) + 2\left(\frac{3}{2}\right) + (2)\right)$

$= \frac{1}{4}(0 + 1 + 2 + 3 + 2)$

$= \frac{8}{4} = 2$

In Exercises 1–6, (a) use the Trapezoidal Rule with $n = 4$ to approximate the value of the integral. (b) Use the concavity of the function to predict whether the approximation is an overestimate or an underestimate. Finally, (c) find the integral's exact value to check your answer.

1. $\int_0^2 x\,dx$ **(a)** 2 **(b)** Exact **(c)** 2

2. $\int_0^2 x^2\,dx$ **(a)** 2.75 **(b)** Over **(c)** $\frac{8}{3}$

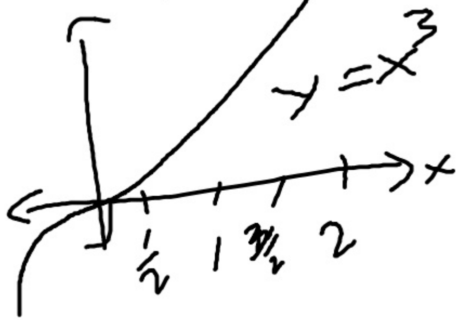
3. $\int_0^2 x^3\,dx$ **(a)** 4.25 **(b)** Over **(c)** 4

4. $\int_1^2 \frac{1}{x}\,dx$ **(a)** 0.697 **(b)** Over **(c)** $\ln 2 \approx 0.693$

5. $\int_0^4 \sqrt{x}\,dx$ **(a)** 5.146 **(b)** Under **(c)** $\frac{16}{3}$

6. $\int_0^{\pi} \sin x\,dx$ **(a)** 1.896 **(b)** Under **(c)** 2

③ $\frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$



x	y	
0	0	y_0
1/2	1/8	y_1
1	1	y_2
3/2	3.375	y_3
2	8	y_4

3

$$T = \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right)$$

$$= \frac{\left(\frac{1}{2}\right)}{2}\left(0 + 2\left(\tfrac{1}{8}\right) + 2(1) + 2(3.375) + 8\right)$$

$$\left(\tfrac{1}{4} + 2 + 6.75 + 8\right)$$

$$= \frac{1}{4}\left(\tfrac{1}{4} + 2 + 6.75 + 8\right)$$

add... multiply...

EXAMPLE 2 Averaging Temperatures

An observer measures the outside temperature every hour from noon until midnight, recording the temperatures in the following table.

Time	N	1	2	3	4	5	6	7	8	9	10	11	M
Temp	63	65	66	68	70	69	68	68	65	64	62	58	55

What was the average temperature for the 12-hour period?

BUT we already have the data!

SOLUTION

We are looking for the average value of a continuous function (temperature) for which we know values at discrete times that are one unit apart. We need to find

$$av(f) = \frac{1}{b-a}\boxed{\int_a^b f(x)\,dx},$$

you don't have f(x)..

without having a formula for $f(x)$. The integral, however, can be approximated by the Trapezoidal Rule, using the temperatures in the table as function values at the points of a 12-subinterval partition of the 12-hour interval (making $h = 1$).

$$T = \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{11} + y_{12}\right)$$

$$= \frac{1}{2}\left(63 + 2 \cdot 65 + 2 \cdot 66 + \cdots + 2 \cdot 58 + 55\right)$$

$$= 782$$

Using T to approximate $\int_a^b f(x)\,dx$, we have

$$av(f) \approx \frac{1}{b-a} \cdot T = \frac{1}{12} \cdot 782 \approx 65.17.$$

Rounding to be consistent with the data given, we estimate the average temperature as 65 degrees. *Now try Exercise 7.*

7. Use the function values in the following table and the Trapezoidal Rule with $n = 6$ to approximate $\int_0^6 f(x)\,dx$.

x	0	1	2	3	4	5	6
$f(x)$	12	10	9	11	13	16	18

7. $\frac{1}{2}(12 + 2(10) + 2(9) + 2(11) + 2(13) + 2(16) + 18) = 74$

8. Use the function values in the following table and the Trapezoidal Rule with $n = 6$ to approximate $\int_2^8 f(x)\,dx$.

x	2	3	4	5	6	7	8
$f(x)$	16	19	17	14	13	16	20

8. $\frac{1}{2}(16 + 2(19) + 2(17) + 2(14) + 2(13) + 2(16) + 20) = 97$

9. ***Volume of Water in a Swimming Pool*** A rectangular swimming pool is 30 ft wide and 50 ft long. The table below shows the depth $h(x)$ of the water at 5-ft intervals from one end of the pool to the other. Estimate the volume of water in the pool using the Trapezoidal Rule with $n = 10$, applied to the integral

$$V = \int_0^{50} 30 \cdot h(x)\,dx.$$

15,990 ft^3

Position (ft) x	Depth (ft) $h(x)$	Position (ft) x	Depth (ft) $h(x)$
0	6.0	30	11.5
5	8.2	35	11.9
10	9.1	40	12.3
15	9.9	45	12.7
20	10.5	50	13.0
25	11.0		

9 $\frac{h}{2}(y_0 + 2y_1 + \ldots + y_{11})$

$\frac{5}{2}(6 + 2(8.2) + \ldots$

EXPLORATION 1 Area Under a Parabolic Arc

The area A_P of a figure having a horizontal base, vertical sides, and a parabolic top (Figure 5.33) can be computed by the formula

$$A_P = \frac{h}{3}\left(l + 4m + r\right),$$

where h is half the length of the base, l and r are the lengths of the left and right sides, and m is the altitude at the midpoint of the base. This formula, once a profound discovery of ancient geometers, is readily verified today with calculus.

1. Coordinatize Figure 5.33 by centering the base at the origin, as shown in Figure 5.34. Let $y = Ax^2 + Bx + C$ be the equation of the parabola. Using this equation, show that $y_0 = Ah^2 - Bh + C$, $y_1 = C$, and $y_2 = Ah^2 + Bh + C$.

2. Show that $y_0 + 4y_1 + y_2 = 2Ah^2 + 6C$.

3. Integrate to show that the area A_P is
$$\frac{h}{3}(2Ah^2 + 6C).$$

4. Combine these results to derive the formula
$$A_P = \frac{h}{3}\left(y_0 + 4y_1 + y_2\right).$$

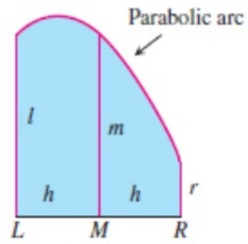


Figure 5.33 The area under the parabolic arc can be computed from the length of the base LR and the lengths of the altitudes constructed at L, R and midpoint M. (Exploration 1)

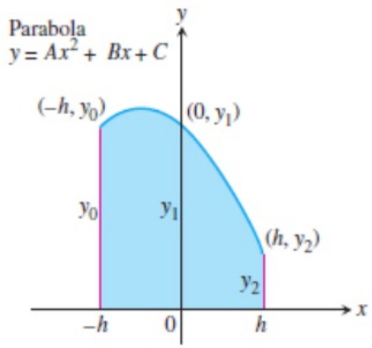


Figure 5.34 A convenient coordinatization of Figure 5.33. The parabola has equation $y = Ax^2 + Bx + C$, and the midpoint of the base is at the origin. (Exploration 1)

parabola, opening down...

This last formula leads to an efficient rule for approximating integrals numerically. Partition the interval of integration into an even number of subintervals, apply the formula for A_P to successive interval pairs, and add the results. This algorithm is known as Simpson's Rule.

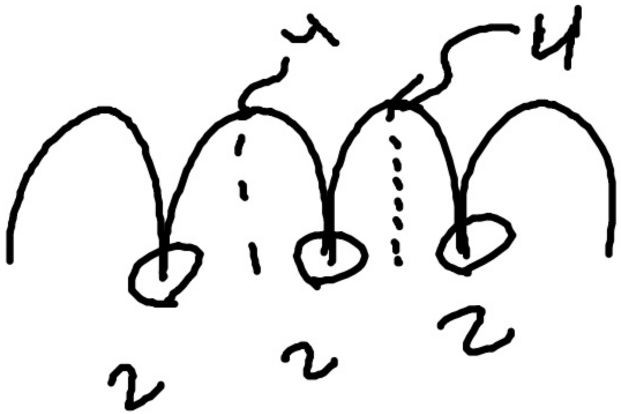
Simpson's Rule

$$A_P = \frac{h}{3}\left(l + 4m + r\right),$$

To approximate $\int_a^b f(x)\,dx$, use

$$S = \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right),$$

where $[a, b]$ is partitioned into an *even* number n of subintervals of equal length $h = (b - a)/n$.



EXAMPLE 3 Applying Simpson's Rule

Use Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4\,dx$.

SOLUTION

Partition $[0, 2]$ into four subintervals and evaluate $y = 5x^4$ at the partition points. (See Table 5.5 on the next page.)

continued

Table 5.5

x	$y = 5x^4$
0	0
$\frac{1}{2}$	$\frac{5}{16}$
1	5
$\frac{3}{2}$	$\frac{405}{16}$
2	80

Then apply Simpson's Rule with $n = 4$ and $h = 1/2$:

$$S = \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right)$$

$$= \frac{1}{6}\left(0 + 4\left(\frac{5}{16}\right) + 2\left(5\right) + 4\left(\frac{405}{16}\right) + 80\right)$$

$$= \frac{385}{12}.$$

This estimate differs from the exact value (32) by only 1/12, a percentage error of less than three-tenths of one percent—and this was with just 4 subintervals.

Now try Exercise 17.

In Exercises 13–18, (a) use Simpson's Rule with $n = 4$ to approximate the value of the integral and (b) find the exact value of the integral to check your answer. (Note that these are the same integrals as Exercises 1–6, so you can also compare it with the Trapezoidal Rule approximation.)

13. $\int_0^2 x\,dx$ 14. $\int_0^2 x^2\,dx$

15. $\int_0^2 x^3\,dx$ 16. $\int_1^2 \frac{1}{x}\,dx$

17. $\int_0^4 \sqrt{x}\,dx$ 18. $\int_0^{\pi} \sin x\,dx$

16

$\Delta x = \frac{b-a}{n}$

step one: find Δx

$\frac{2-1}{4} = \frac{1}{4} = h$

$y_0 \Leftarrow 1$
$y_1 \Leftarrow 1\frac{1}{4}$
$y_2 \Leftarrow 1\frac{1}{2}$
$y_3 \Leftarrow 1\frac{3}{4}$
$y_4 \Leftarrow 2$

step 2: find x values

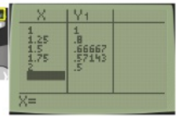
step 3: plug in x values into the function

a) 2nd→TBLSET

b) select "Ask" for Indpnt:

c) 2nd →TABLE

d) plug in values!



step 4: plug values into Simpson's Rule formula.

$$S = \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right)$$

$$S = \frac{\frac{1}{4}}{3}\left(1 + 4(.8) + 2\left(\tfrac{2}{3}\right) + 4(.57) + .5\right)$$

$$= \frac{1}{12}\left(1 + 3.2 + \tfrac{4}{3} + 4(.57) + .5\right)$$

Use this same approach for Trapazoidal rule, only use the Trapazoidal Rule instead on step 4!

19. Consider the integral $\int_{-1}^{3} (x^3 - 2x)\, dx$.

(a) Use Simpson's Rule with $n = 4$ to approximate its value.

(b) Find the exact value of the integral. What is the error, $|E_S|$?

(c) Explain how you could have predicted what you found in (b) from knowing the error-bound formula.

(d) **Writing to Learn** Is it possible to make a general statement about using Simpson's Rule to approximate integrals of cubic polynomials? Explain.