What you'll learn about

- Differential Equations
- Slope Fields
- Euler's Method

. . . and why

Differential equations have always been a prime motivation for the study of calculus and remain so to this day.

6.1 Slope Fields and Euler's Method

We will be practicing a lot of things that we've already done (antiderivatives), but now we're becoming more efficient at it.

Also, we are going to learn how to graph antiderivatives.

Kinda. $\int f(x) = F(x) + \underline{C}$

DEFINITION Differential Equation

An equation involving a derivative is called a **differential equation.** The **order of a differential equation** is the order of the highest derivative involved in the equation.

EXAMPLE 1 Solving a Differential Equation

Find all functions y that satisfy $dy/dx = \sec^2 x + 2x + 5$.

$$\frac{d}{dx}\left(\tan x\right) = \sec^2 x$$

SOLUTION

We first encountered this sort of differential equation (called *exact* because it gives the derivative exactly) in Chapter 4. The solution can be any antiderivative of $\sec^2 x + 2x + 5$, which can be any function of the form $y = \tan x + x^2 + 5x + C$. That family of functions is the *general* solution to the differential equation.

Now try Exercise 1.

Your task for the first set of assigned problems will be to find the *general* solution, then you will need to find the value of the constant C to find the *particular* solution.

Notice that we cannot find a unique solution to a differential equation unless we are given further information. If the general solution to a first-order differential equation is continuous, the only additional information needed is the value of the function at a single point, called an *initial condition*. A differential equation with an initial condition is called an *initial value problem*. It has a unique solution, called the *particular solution* to the differential equation.

This will help us find "C"!

In Exercises 1–10, find the general solution to the exact differential equation.

1. $\dfrac{dy}{dx} = 5x^4 - \sec^2 x$ $\quad y = x^5 - \tan x + C$

2. $\dfrac{dy}{dx} = \sec x \tan x - e^x$ $\quad y = \sec x - e^x + C$

3. $\dfrac{dy}{dx} = \sin x - e^{-x} + 8x^3$ $\quad y = -\cos x + e^{-x} + 2x^4 + C$

4. $\dfrac{dy}{dx} = \dfrac{1}{x} - \dfrac{1}{x^2}\ (x > 0)$ $\quad y = \ln x + x^{-1} + C$

5. $\dfrac{dy}{dx} = 5^x \ln 5 + \dfrac{1}{x^2 + 1}$ $\quad y = 5^x + \tan^{-1} x + C$

6. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2}} - \dfrac{1}{\sqrt{x}}$ $\quad y = \sin^{-1} x - 2\sqrt{x} + C$

7. $\dfrac{dy}{dt} = 3t^2 \cos(t^3)$ $\quad y = \sin(t^3) + C$

8. $\dfrac{dy}{dt} = (\cos t)\, e^{\sin t}$ $\quad y = e^{\sin t} + C$

9. $\dfrac{du}{dx} = (\sec^2 x^5)(5x^4)$ $\quad u = \tan(x^5) + C$

10. $\dfrac{dy}{du} = 4(\sin u)^3(\cos u)$ $\quad y = (\sin u)^4 + C$

$g'(f(x)) \cdot f'(x) + C$ $\quad x^{-2}$

$x^5 - \tan x + C$

$\frac{1}{x} = x$

$\frac{d}{dx}[\sec x] = \sec x \tan x$

$\sin t^3$

$\frac{d}{dx}[\sin x] = \cos x$

$\frac{d}{dx}(\ln x) = \frac{1}{x}$

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1 - x^2}}$$

EXAMPLE 2 Solving an Initial Value Problem

Find the particular solution to the equation $dy/dx = e^x - 6x^2$ whose graph passes through the point (1, 0).

x y

SOLUTION

The general solution is $y = e^x - 2x^3 + C$. Applying the initial condition, we have $0 = e - 2 + C$, from which we conclude that $C = 2 - e$. Therefore, the particular solution is $y = e^x - 2x^3 + 2 - e$.

Now try Exercise 13.

$$\frac{dy}{dx} \to y$$

$$y = e^x - 2x^3 + C$$

$$(0) = e^1 - 2(1)^3 + C$$

$$(0) = e - 2 + C$$

① Antiderive

② Plug in "initial conditions"

③ Find C, write particular solution!

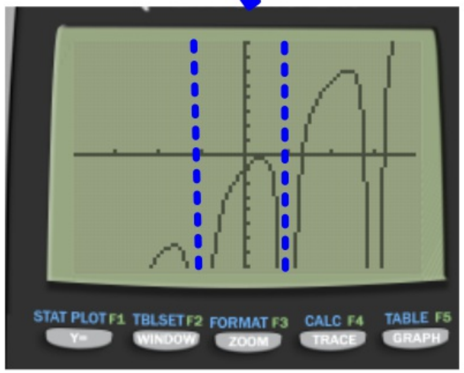
EXAMPLE 3 Handling Discontinuity in an Initial Value Problem

Find the particular solution to the equation $dy/dx = 2x - \sec^2 x$ whose graph passes through the point $(0, 3)$.

SOLUTION

The general solution is $y = x^2 - \tan x + C$. Applying the initial condition, we have $3 = 0 - 0 + C$, from which we conclude that $C = 3$. Therefore, the particular solution is $y = x^2 - \tan x + 3$. Since the point $(0, 3)$ only pins down the continuous piece of the general solution over the interval $\boxed{(-\pi/2, \pi/2),}$ we add the domain stipulation $-\pi/2 < x < \pi/2$.

Now try Exercise 15.



This is the graph of dy/dx. Notice that the interval is the result from the given point (0,3), since 0 is in between the interval of the continuous graph.

In Exercises 11–20, solve the initial value problem explicitly.

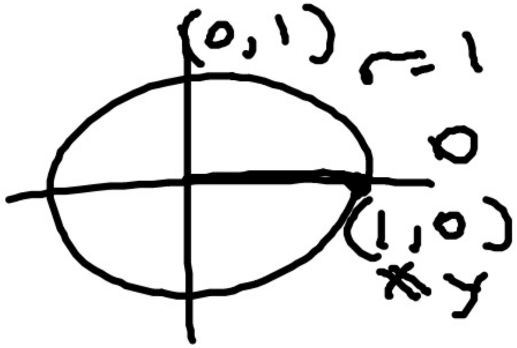
11. $\dfrac{dy}{dx} = 3 \sin x$ and $y = 2$ when $x = 0$ $\quad y = -3 \cos x + 5$

$y = -3\cos x + C$

$2 = -3\cos(0) + C$

$2 = -3(1) + C$

$C = 5$



19. $\dfrac{dv}{dt} = 4 \sec t \tan t + e^t + 6t$ and $v = 5$ when $t = 0$

$v = 4 \sec t + e^t + 3t^2$ $(-\pi/2 < t < \pi/2)$ (Note that $C = 0$.)

$\sec\theta = \dfrac{r}{x}$

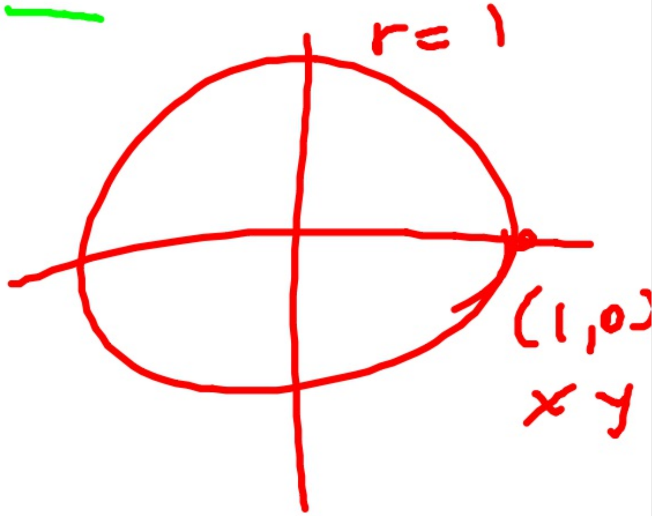
$v = 4\sec t + e^t + 3t^2 + C$

$5 = 4\sec(0) + e^0 + 3(0)^2 + C$

$5 = 4(1) + 1 + C$

$5 = 5 + C$

$C = 0$



12. $\frac{dy}{dx} = 2e^x - \cos x$ and $y = 3$ when $x = 0$ $\quad y = 2e^x - \sin x + 1$

$= 2e^x - \sin x + C$

$3 = 2e^0 - \sin(0) + C$

$3 = 2(1) - 0 + C$

$C = 1$

$y = 2e^x - \sin x + 1$

13. $\dfrac{du}{dx} = 7x^6 - 3x^2 + 5$ and $u = 1$ when $x = 1$ $u = x^7 - x^3 + 5x - 4$

$= x^7 - x^3 + 5x + C$

$1 = 1^7 - 1^3 + 5(1) + C$

$C = -4$

$u = x^7 - x^3 + 5x - 4$

14. $\frac{dA}{dx} = 10x^9 + 5x^4 - 2x + 4$ and $A = 6$ when $x = 1$

$A = x^{10} + x^5 - x^2 + 4x + 1$

$= x^{10} + x^5 - x^2 + 4x + C$

$6 = (1)^{10} + (1)^5 - (1)^2 + 4(1) + C$

$6 = 5 + C$

$C = 1$

$A = x^{10} + x^5 - x^2 + 4x + 1$

15. $\frac{dy}{dx} = -\frac{1}{x^2} - \frac{3}{x^4} + 12$ and $y = 3$ when $x = 1$

$y = x^{-1} + x^{-3} + 12x - 11 \ (x > 0)$

$y = -x^{-2} - 3 \cdot x^{-4}$

$y = x^{-1} + x^{-3} + 12x + C$

$3 = (1)^{-1} + (1)^{-3} + 12(1) + C$

$3 = 1 + 1 + 12 + C$

$3 = 14 + C$

$-11 = C$

$y =$

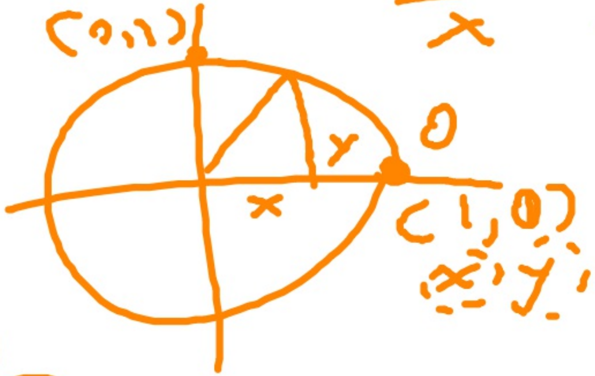
$y = x \ldots + x \ldots + 12x - 11\ (x > 0)$

16. $\dfrac{dy}{dx} = 5\sec^2 x - \dfrac{3}{2}\sqrt{x}$ and $y = 7$ when $x = 0$

$y = 5\tan x - x^{3/2} + 7\ (0 < x < \pi/2)$

$y = 5\tan x - x^{3/2} + C$

$\tan\theta = \dfrac{y}{x} = \dfrac{0}{1}$



$7 = 5(0) - 0 + C$

$7 = C$

$y = 5\tan x - x^{3/2} + 7$

In Exercises 11–20, solve the initial value problem explicitly.

11. $\dfrac{dy}{dx} = 3\sin x$ and $y = 2$ when $x = 0$ $\quad y = -3\cos x + 5$

12. $\dfrac{dy}{dx} = 2e^x - \cos x$ and $y = 3$ when $x = 0$ $\quad y = 2e^x - \sin x + 1$

13. $\dfrac{du}{dx} = 7x^6 - 3x^2 + 5$ and $u = 1$ when $x = 1$ $\quad u = x^7 - x^3 + 5x - 4$

14. $\dfrac{dA}{dx} = 10x^9 + 5x^4 - 2x + 4$ and $A = 6$ when $x = 1$

$A = x^{10} + x^5 - x^2 + 4x + 1$

15. $\dfrac{dy}{dx} = -\dfrac{1}{x^2} - \dfrac{3}{x^4} + 12$ and $y = 3$ when $x = 1$

$y = x^{-1} + x^{-3} + 12x - 11\ (x > 0)$

16. $\dfrac{dy}{dx} = 5\sec^2 x - \dfrac{3}{2}\sqrt{x}$ and $y = 7$ when $x = 0$

$y = 5\tan x - x^{3/2} + 7\ (0 < x < \pi/2)$

17. $\dfrac{dy}{dt} = \dfrac{1}{1 + t^2} + 2^t \ln 2$ and $y = 3$ when $t = 0$

$y = \tan^{-1} t + 2^t + 2$

18. $\dfrac{dx}{dt} = \dfrac{1}{t} - \dfrac{1}{t^2} + 6$ and $x = 0$ when $t = 1$

$x = \ln t + t^{-1} + 6t - 7\ (t > 0)$

19. $\dfrac{dv}{dt} = 4\sec t\tan t + e^t + 6t$ and $v = 5$ when $t = 0$

$v = 4\sec t + e^t + 3t^2\ (-\pi/2 < t < \pi/2)$ (Note that $C = 0$.)

20. $\dfrac{ds}{dt} = t(3t - 2)$ and $s = 0$ when $t = 1$

$s = t^3 - t^2$ (Note that $C = 0$.)

EXAMPLE 4 Using the Fundamental Theorem to Solve an Initial Value Problem

Find the solution to the differential equation $f'(x) = e^{-x^2}$ for which $\boxed{f(7) = 3.}$

"initial condition."

SOLUTION

answer!

This almost seems too simple, but $\boxed{f(x) = \int_7^x e^{-t^2}\,dt + 3}$ has both of the necessary properties! Clearly, $f(7) = \int_7^7 e^{-t^2}\,dt + 3 = 0 + 3 = 3$, and $f'(x) = e^{-x^2}$ by the Fundamental Theorem.

The integral form of the solution in Example 4 might seem less desirable than the explicit form of the solutions in Examples 2 and 3, but (thanks to modern technology) it does enable us to find $f(x)$ for any x. For example, $f(-2) = \int_7^{-2} e^{-t^2}\,dt + 3 =$ fnInt $(e$^$(-t^2), t, 7, -2) + 3 \approx 1.2317$.

Now try Exercise 21.

In Exercises 21–24, solve the initial value problem using the Fundamental Theorem. (Your answer will contain a definite integral.)

21. $\dfrac{dy}{dx} = \sin(x^2)$ and $y = 5$ when $x = 1$

22. $\dfrac{du}{dx} = \sqrt{2 + \cos x}$ and $u = -3$ when $x = 0$ $u = \int_0^x \sqrt{2 + \cos t}\, dt - 3$

23. $F'(x) = e^{\cos x}$ and $F(2) = 9$

24. $G'(s) = \sqrt[3]{\tan s}$ and $G(0) = 4$

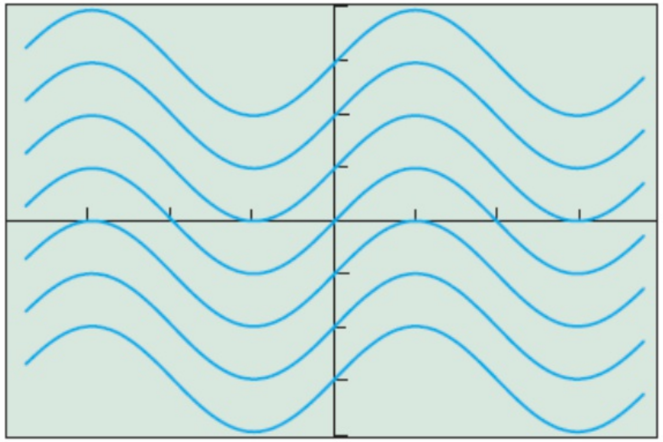
EXAMPLE 5 Graphing a General Solution

Graph the family of functions that solve the differential equation $dy/dx = \cos x$.

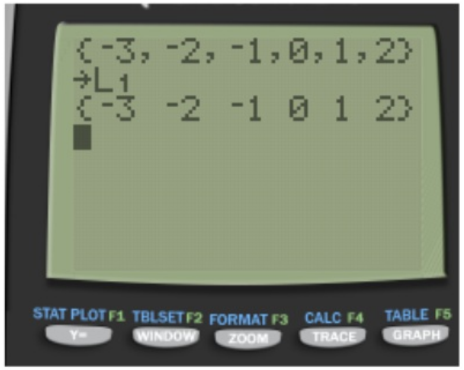
SOLUTION

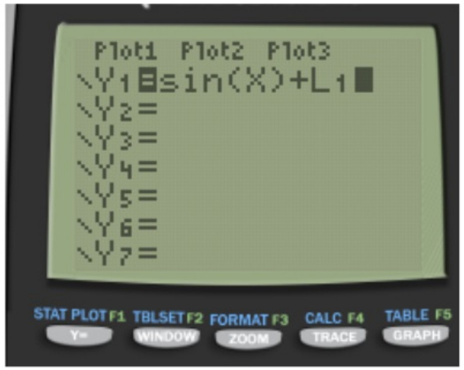
Any function of the form $y = \sin x + C$ solves the differential equation. We cannot graph them all, but we can graph enough of them to see what a family of solutions would look like. The command $\{-3, -2, -1, 0, 1, 2, 3,\} \to L_1$ stores seven values of C in the list L_1. Figure 6.1 shows the result of graphing the function $Y_1 = \sin(x) + L_1$.

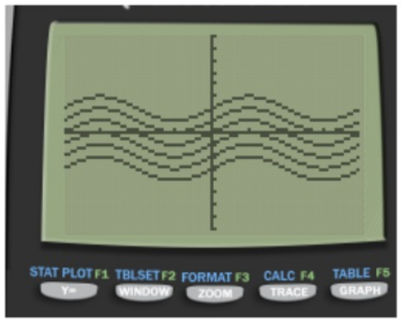
Now try Exercises 25–28.



[–2π, 2π] by [–4, 4]

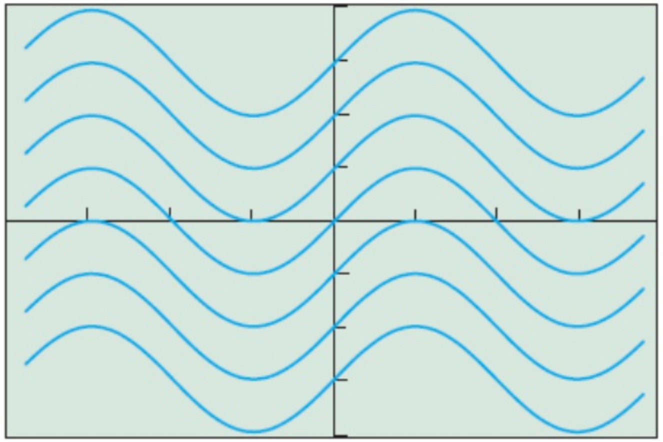






look how the information was put in.
what are we looking at when we graph it?

Notice that the graph in Figure 6.1 consists of a family of parallel curves. This should come as no surprise, since functions of the form $\sin(x) + C$ are all vertical translations of the basic sine curve. It might be less obvious that we could have predicted the appearance of this family of curves from *the differential equation itself*. Exploration 1 gives you a new way to look at the solution graph.



$[-2\pi, 2\pi]$ by $[-4, 4]$

Figure 6.1 A graph of the family of functions $Y_1 = \sin(x) + L_1$, where $L_1 = \{-3, -2, -1, 0, 1, 2, 3\}$. This graph shows some of the functions that satisfy the differential equation $dy/dx = \cos x$. (Example 5)

We will be constructing slope fields

Bad news; we don't have this program in our TI graphing calculator, BUT it can be uploaded (I'll leave that up to you).

Good news; desmos has slope fields set up online, we just need to search for it...google "slope fields desmos."

desmos slope fields

EXAMPLE 6 Constructing a Slope Field

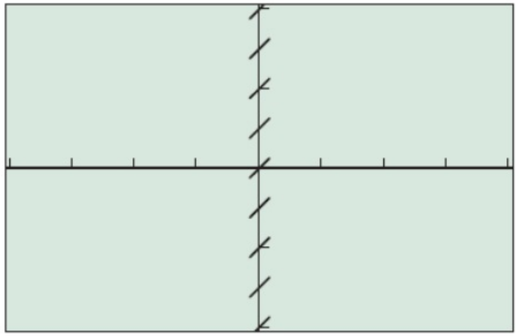
Construct a slope field for the differential equation $dy/dx = \cos x$.

use integers
for "y"!

SOLUTION

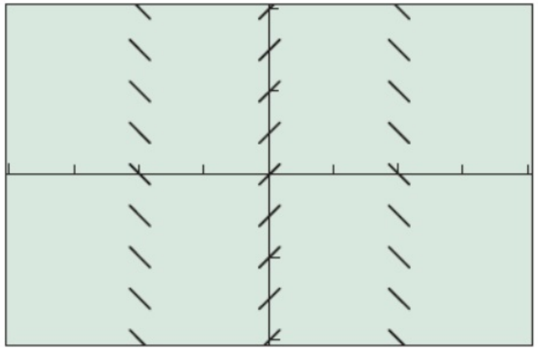
We know that the slope at any point $(0, y)$ will be $\cos 0 = 1$, so we can start by drawing tiny segments with slope 1 at several points along the y-axis (Figure 6.2a). Then, since the slope at any point (π, y) or $(-\pi, y)$ will be -1, we can draw tiny segments with slope -1 at several points along the vertical lines $x = \pi$ and $x = -\pi$ (Figure 6.2b). The slope at all odd multiples of $\pi/2$ will be zero, so we draw tiny horizontal segments along the lines $x = \pm\pi/2$ and $x = \pm 3\pi/2$ (Figure 6.2c). Finally, we add tiny segments of slope 1 along the lines $x = \pm 2\pi$ (Figure 6.2d).

Now try Exercise 29.



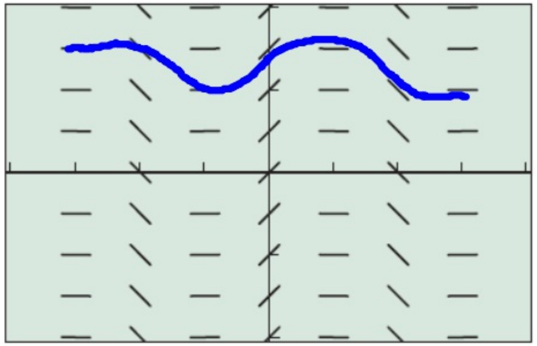
$[-2\pi, 2\pi]$ by $[-4, 4]$

(a)



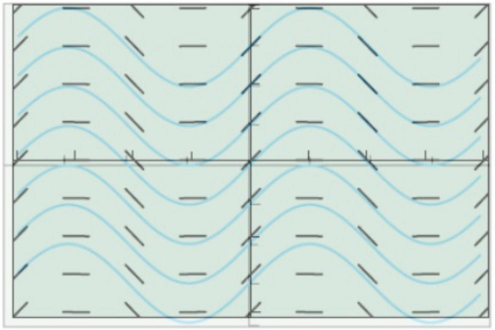
$[-2\pi, 2\pi]$ by $[-4, 4]$

(b)



$[-2\pi, 2\pi]$ by $[-4, 4]$

(c)



$[-2\pi, 2\pi]$ by $[-4, 4]$

(d)

Figure 6.1 A graph of the family of

Convince yourself that we drew the graph of the antiderivative using values of slopes!

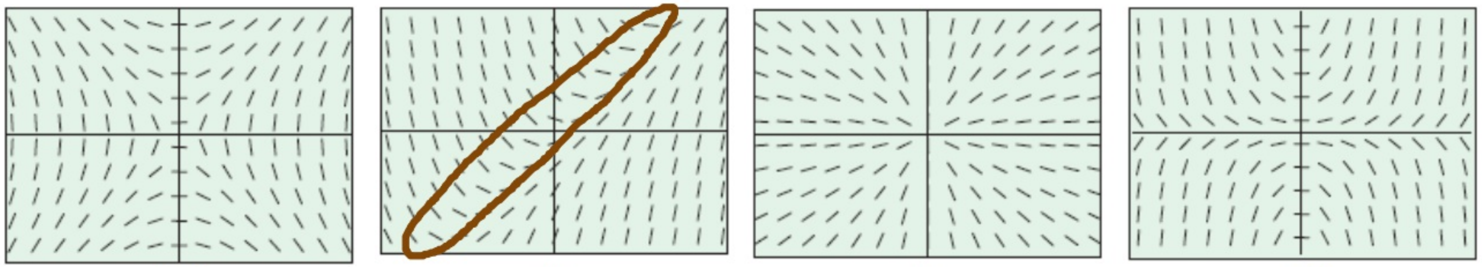
We must do this by hand

EXAMPLE 8 Matching Slope Fields with Differential Equations

Use slope analysis to match each of the following differential equations with one of the slope fields (a) through (d). (Do not use your graphing calculator.)

Let's practice!

1. $\dfrac{dy}{dx} = x - y$ 2. $\dfrac{dy}{dx} = xy$ 3. $\dfrac{dy}{dx} = \dfrac{x}{y}$ 4. $\dfrac{dy}{dx} = \dfrac{y}{x}$



(a) (b) (c) (d)

① cont.

let $\dfrac{dy}{dx} = 0$

$0 = x - y$

$+y \quad +y$

$y = x$

SOLUTION

To match Equation 1, we look for a graph that has zero slope along the line $x - y = 0$. That is graph (b).

To match Equation 2, we look for a graph that has zero slope along both axes. That is graph (d).

To match Equation 3, we look for a graph that has horizontal segments when $x = 0$ and vertical segments when $y = 0$. That is graph (a).

To match Equation 4, we look for a graph that has vertical segments when $x = 0$ and horizontal segments when $y = 0$. That is graph (c).

Now try Exercise 39.

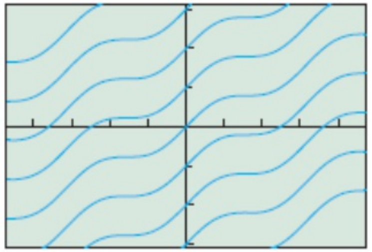
In Exercises 25–28, match the differential equation with the graph of a family of functions (a)–(d) that solve it. Use slope analysis, not your graphing calculator.

25. $\dfrac{dy}{dx} = (\sin x)^2$ Graph (b)

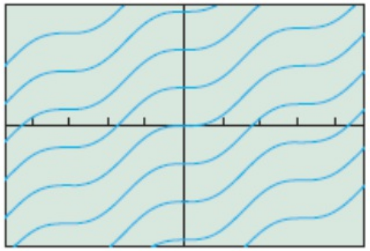
26. $\dfrac{dy}{dx} = (\sin x)^3$ Graph (c)

27. $\dfrac{dy}{dx} = (\cos x)^2$ Graph (a)

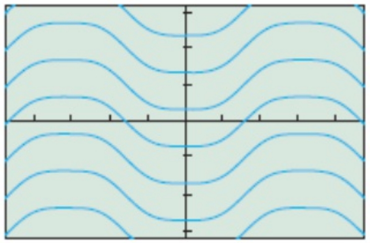
28. $\dfrac{dy}{dx} = (\cos x)^3$ Graph (d)



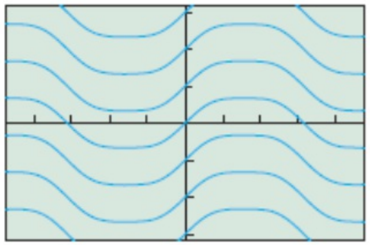
(a)



(b)

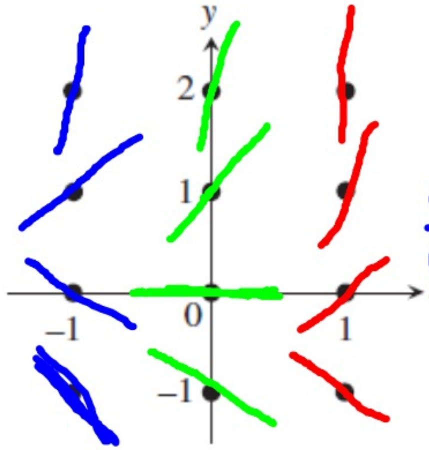


(c)



(d)

In Exercises 29–34, construct a slope field for the differential equation. In each case, copy the graph at the right and draw tiny segments through the twelve lattice points shown in the graph. Use slope analysis, not your graphing calculator.



This is a good way to draw slope fields!

$\frac{dy}{dx} = x + 2y$

33

x	y	$\frac{dy}{dx}$
-1	-1	-3
-1	0	-1
-1	1	1
-1	2	3

29. $\frac{dy}{dx} = x$ **30.** $\frac{dy}{dx} = y$ **31.** $\frac{dy}{dx} = 2x + y$

32. $\frac{dy}{dx} = 2x - y$ **33.** $\frac{dy}{dx} = x + 2y$ **34.** $\frac{dy}{dx} = x - 2y$

x	y	$\frac{dy}{dx}$
0	-1	-2
0	0	0
0	1	2
0	2	4

x	y	$\frac{dy}{dx}$
1	-1	-1
1	0	1
1	1	3
1	2	5

EXAMPLE 7 Constructing a Slope Field for a Nonexact Differential Equation

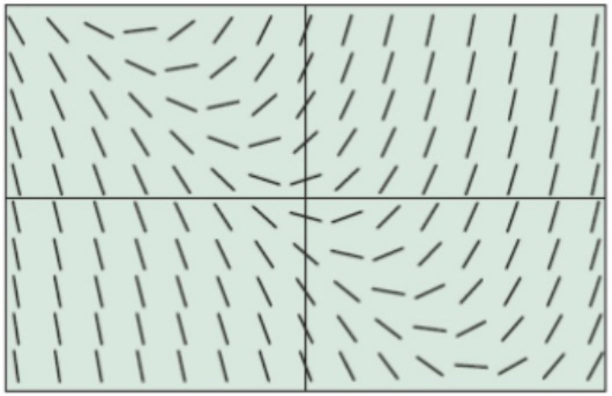
Use a calculator to construct a slope field for the differential equation $dy/dx = x + y$ and sketch a graph of the particular solution that passes through the point (2, 0).

SOLUTION

The calculator produces a graph like the one in Figure 6.5a. Notice the following properties of the graph, all of them easily predictable from the differential equation:

1. The slopes are zero along the line $x + y = 0$.
2. The slopes are -1 along the line $x + y = -1$.
3. The slopes get steeper as x increases.
4. The slopes get steeper as y increases.

These are ways to fill in the slope field a little faster...



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

(a)

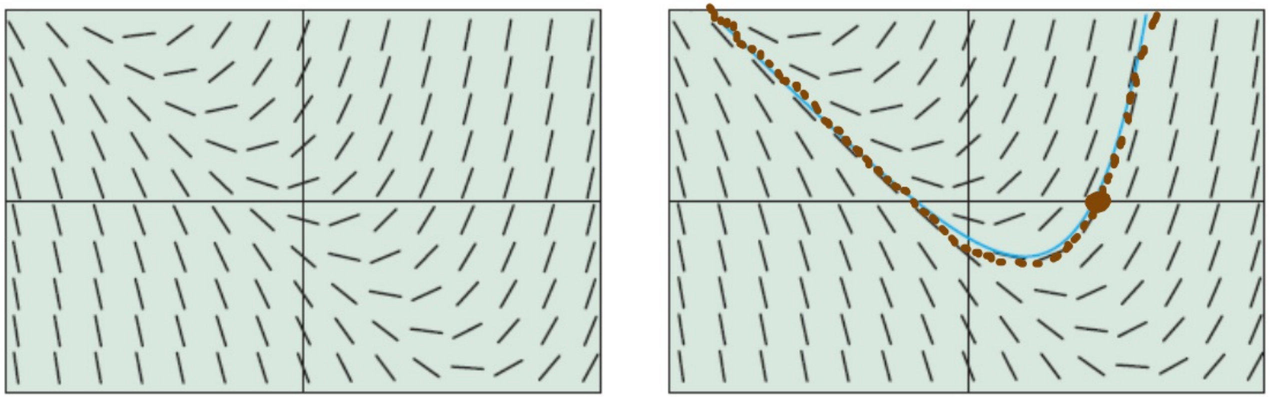
Just use integers, unlike this graph...

EXAMPLE 7 Constructing a Slope Field for a Nonexact Differential Equation

(cont.)

Use a calculator to construct a slope field for the differential equation $dy/dx = x + y$ and sketch a graph of the particular solution that passes through the point (2, 0).

The particular solution can be found by drawing a smooth curve through the point (2, 0) that follows the slopes in the slope field, as shown in Figure 6.5b.



[−4.7, 4.7] by [−3.1, 3.1]
(a)

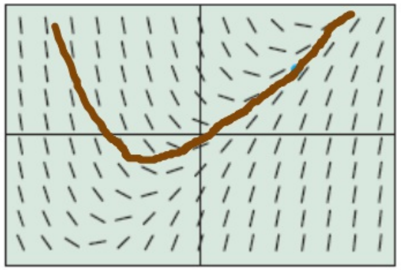
[−4.7, 4.7] by [−3.1, 3.1]
(b)

Mark where (2,0) is on the graph, then use the slope field to guide your graph.

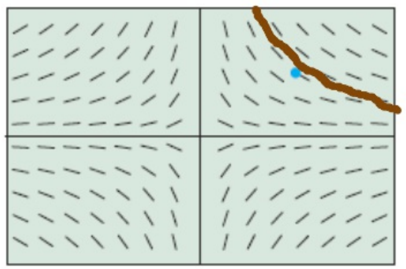
They are slopes, after all...

Figure 6.5 (a) A slope field for the differential equation $dy/dx = x + y$, and (b) the same slope field with the graph of the particular solution through (2, 0) superimposed. (Example 7)

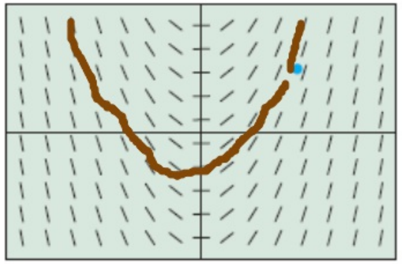
In Exercises 35–40, match the differential equation with the appropriate slope field. Then use the slope field to sketch the graph of the particular solution through the highlighted point (3, 2). (All slope fields are shown in the window [−6, 6] by [−4, 4].)



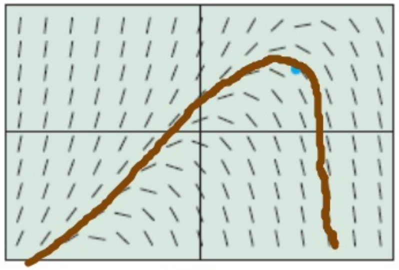
(a)



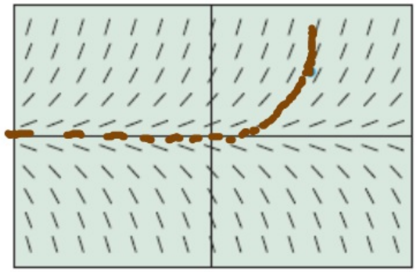
(b)



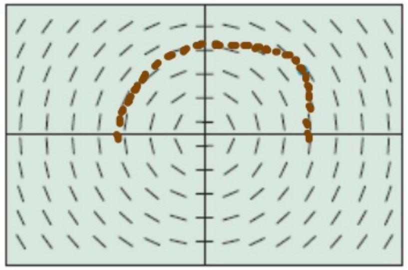
(c)



(d)



(e)



(f)

35. $\dfrac{dy}{dx} = x$ C

36. $\dfrac{dy}{dx} = y$

37. $\dfrac{dy}{dx} = x - y$

38. $\dfrac{dy}{dx} = y - x$

39. $\dfrac{dy}{dx} = -\dfrac{y}{x}$

40. $\dfrac{dy}{dx} = -\dfrac{x}{y}$