What you'll learn about

- Indefinite Integrals
- Leibniz Notation and Antiderivatives
- Substitution in Indefinite Integrals
- Substitution in Definite Integrals

. . . and why

Antidifferentiation techniques were historically crucial for applying the results of calculus.

6.2 Antidifferentiation by Substitution

As mentioned before, our focus will be to learn new tricks to help us integrate.

First, we will accept a mathamatical symbol that will tell us to perform the antiderivative.

DEFINITION Indefinite Integral

The family of all antiderivatives of a function $f(x)$ is the **indefinite integral of f with respect to x** and is denoted by $\int f(x)dx$.

If F is any function such that $F'(x) = f(x)$, then $\int f(x)dx = F(x) + C$, where C is an arbitrary constant, called the **constant of integration.**

As in Chapter 5, the symbol $\int$ is an **integral sign,** the function f is the **integrand** of the integral, and x is the **variable of integration.**

EXAMPLE 1 Evaluating an Indefinite Integral

Evaluate $\int (x^2 - \sin x)\, dx$.

SOLUTION

Evaluating this definite integral is just like solving the differential equation $dy/dx = x^2 - \sin x$. Our past experience with derivatives leads us to conclude that

$$\int (x^2 - \sin x)\, dx = \frac{x^3}{3} + \cos x + C$$

(as you can check by differentiating).

Now try Exercise 3.

In Exercises 1–6, find the indefinite integral.

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

1. $\int (\cos x - 3x^2)\, dx$ $\quad \sin x - x^3 + C$

2. $\int x^{-2}\, dx$ $\quad -x^{-1} + C$

$t^2 - t^{-2}$

3. $\int \left(t^2 - \frac{1}{t^2}\right) dt$ $\quad t^3/3 + t^{-1} + C$

4. $\int \frac{dt}{t^2+1}$ $\quad \tan^{-1} t + C$

5. $\int (3x^4 - 2x^{-3} + \sec^2 x)\, dx$ $\quad (3/5)x^5 + x^{-2} + \tan x + C$

6. $\int (2e^x + \sec x \tan x - \sqrt{x})\, dx$ $\quad 2e^x + \sec x - (2/3)x^{3/2} + C$

Properties of Indefinite Integrals

$$\int k\,f(x)dx = k\int f(x)\,dx \quad \text{for any constant } k$$

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Power Formulas

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C \text{ when } n \neq -1$$

$$\int u^{-1}\,du = \int \frac{1}{u}\,du = \ln|u| + C$$

(see Example 2)

Trigonometric Formulas

$$\int \cos u\,du = \sin u + C$$

$$\int \sin u\,du = -\cos u + C$$

$$\int \sec^2 u\,du = \tan u + C$$

$$\int \csc^2 u\,du = -\cot u + C$$

$$\int \sec u \tan u\,du = \sec u + C$$

$$\int \csc u \cot u\,du = -\csc u + C$$

Exponential and Logarithmic Formulas

$$\int e^u\,du = e^u + C$$

$$\int a^u\,du = \frac{a^u}{\ln a} + C$$

$$\int \ln u\,du = u\ln u - u + C \quad \text{(See Example 2)}$$

$$\int \log_a u\,du = \int \frac{\ln u}{\ln a}\,du = \frac{u\ln u - u}{\ln a} + C$$

Remember that cheat sheet I gave you?

Now, do all of those *in reverse.*

That's what is being shown on the left...

EXAMPLE 2 Verifying Antiderivative Formulas

Verify the antiderivative formulas:

(a) $\int u^{-1}\,du = \int \frac{1}{u}\,du = \ln|u| + C$ (b) $\int \ln u\,du = u \ln u - u + C$

SOLUTION

We can verify antiderivative formulas by differentiating.

(a) For $u > 0$, we have $\frac{d}{du}(\ln|u| + C) = \frac{d}{du}(\ln u + C) = \frac{1}{u} + 0 = \frac{1}{u}$.

For $u < 0$, we have $\frac{d}{du}(\ln|u| + C) = \frac{d}{du}(\ln(-u) + C) = \frac{1}{-u}(-1) + 0 = \frac{1}{u}$.

Since $\frac{d}{du}(\ln|u| + C) = \frac{1}{u}$ in either case, $\ln|u| + C$ is the general antiderivative of the function $\frac{1}{u}$ on its entire domain.

product rule!

(b) $\frac{d}{du}(u \ln u - u + C) = 1 \cdot \ln u + u\left(\frac{1}{u}\right) - 1 + 0 = \ln u + 1 - 1 = \ln u.$

Now try Exercise 11.

In Exercises 7–12, use differentiation to verify the antiderivative formula.

7. $\int \csc^2 u \, du = -\cot u + C$

$(-\cot u + C)' = -(-\csc^2 u) = \csc^2 u$

8. $\int \csc u \cot u = -\csc u + C$

9. $\int e^{2x} \, dx = \frac{1}{2} e^{2x} + C$

10. $\int 5^x \, dx = \frac{1}{\ln 5} 5^x + C$

$\frac{d}{dx} e^u = e^u \cdot u'$

11. $\int \frac{1}{1 + u^2} \, du = \tan^{-1} u + C$

12. $\int \frac{1}{\sqrt{1 - u^2}} \, du = \sin^{-1} u + C$

9 $\frac{d}{dx}\left(\frac{1}{2} e^{2x} + C\right)$

$= \frac{1}{2} e^{2x} \cdot 2 = e^{2x}$

EXAMPLE 3 Paying Attention to the Differential

Let $f(x) = x^3 + 1$ and let $u = x^2$. Find each of the following antiderivatives in terms of x:

(a) $\int f(x)\,dx$ **(b)** $\int f(u)\,du$ **(c)** $\int f(u)\,dx$

SOLUTION

(a) $\displaystyle\int f(x)\,dx = \int (x^3 + 1)\,dx = \frac{x^4}{4} + x + C$

① plug in "u" ② Antiderive ③ plug in "x"

(b) $\displaystyle\int f(u)\,du = \int (u^3 + 1)\,du = \frac{u^4}{4} + u + C = \frac{(x^2)^4}{4} + x^2 + C = \frac{x^8}{4} + x^2 + C$

(c) $\displaystyle\int f(u)\,dx = \int (u^3 + 1)\,dx = \int ((x^2)^3 + 1)\,dx = \int (x^6 + 1)\,dx = \frac{x^7}{7} + x + C$

① plug in "u" ② plug in "x" ③ antiderive

Now try Exercise 15.

In Exercises 13–16, verify that $\boxed{\int f(u)\,du} \neq \boxed{\int f(u)\,dx}$

13. $f(u) = \sqrt{u}$ and $u = x^2\ (x > 0)$

14. $f(u) = u^2$ and $u = x^5$

15. $f(u) = e^u$ and $u = 7x$

16. $f(u) = \sin u$ and $u = 4x$

$\frac{1}{2} + 1$

13

$\int f(u)\,du$

$\int \sqrt{u}\,du = \int u^{\frac{1}{2}}\,du$

$= \frac{2}{3} u^{3/2} + C$

$= \frac{2}{3} (x^2)^{3/2} + C$

$= \boxed{\frac{2}{3} x^3 + C}$

$\int f(u)\,dx$.

$= \int \sqrt{u}\,dx$

$\int \sqrt{x^2}\,dx$

$\int x\,dx$

$\boxed{= \frac{1}{2} x^2 + C}$

$\neq$

Substitution in Indefinite Integrals

A change of variables can often turn an unfamiliar integral into one that we can evaluate. The important point to remember is that it is *not sufficient* to change an integral of the form $\int f(x)\,dx$ into an integral of the form $\int g(u)\,dx$. The differential matters. A complete substitution changes the integral $\int f(x)\,dx$ into an integral of the form $\int g(u)\,du$.

EXAMPLE 4 Using Substitution

Evaluate $\int \sin x\, e^{\cos x}\, dx$.

chain rule in reverse...

SOLUTION

Let $u = \cos x$. Then $du/dx = -\sin x$, from which we conclude that $du = -\sin x\, dx$. We rewrite the integral and proceed as follows:

check...

$$\begin{aligned}
\int \sin x\, e^{\cos x}\, dx &= -\int (-\sin x) e^{\cos x}\, dx \\
&= -\int e^{\cos x} \cdot (-\sin x) dx \\
&= -\int e^{u}\, du \\
&= -e^{u} + C \\
&= -e^{\cos x} + C
\end{aligned}$$

$$\frac{d}{dx}\left(-e^{\cos x} + C\right)$$

$u = \cos x$

$u' = -\sin x$

$+e^{\cos x}(+\sin x)$

In Exercises 17–24, use the indicated substitution to evaluate the integral. Confirm your answer by differentiation.

17. $\int \sin 3x\,dx, \quad u = 3x \qquad -\frac{1}{3}\cos 3x + C$

$u = 3x$

$du = 3\,dx$

$\frac{1}{3}\int 3\sin 3x\,dx$

$= \frac{1}{3}\int \sin u \cdot du$

$= \frac{1}{3}(-\cos u) + C$

$= -\frac{1}{3}\cos 3x + C$

18. $\int x\cos(2x^2)\,dx, \quad u = 2x^2$

$\sec u \tan u$

19. $\int \sec 2x \tan 2x\,dx, \quad u = 2x \qquad \frac{1}{2}\sec 2x + C$

20. $\int 28(7x-2)^3\,dx, \quad u = 7x - 2$

21. $\int \frac{dx}{x^2+9}, \quad u = \frac{x}{3} \qquad (1/3)\tan^{-1}(x/3) + C$

22. $\int \frac{9r^2\,dr}{\sqrt{1-r^3}}, \quad u = 1 - r^3$

23. $\int \left(1 - \cos\frac{t}{2}\right)^2 \sin\frac{t}{2}\,dt, \quad u = 1 - \cos\frac{t}{2} \qquad \frac{2}{3}\left(1 - \cos\frac{t}{2}\right)^3 + C$

24. $\int 8(y^4 + 4y^2 + 1)^2(y^3 + 2y)\,dy, \quad u = y^4 + 4y^2 + 1$

$(1/3)\tan^{-1}(x/3) + C$

21. $\displaystyle\int \frac{dx}{x^2+9}, \quad u = \frac{x}{3}$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$u = \frac{x}{3} = \frac{1}{3}x$

$\tan^{-1}(u)$

$du = \frac{1}{3}dx$

23. $\displaystyle\int\left(1-\cos\frac{t}{2}\right)^2\sin\frac{t}{2}\,dt,\quad u=1-\cos\frac{t}{2}$

$$u = 1-\cos\frac{t}{2}$$

$$du = \sin\frac{t}{2}\cdot\frac{1}{2}\,dt$$

$$2\int\frac{1}{2}\left(1-\cos t/2\right)\sin\frac{1}{2}\,dt$$

$$2\int u^1\,du = 2\left(\frac{u^2}{2}\right)$$

$$= u^2$$

$$= \left(1-\cos\frac{t}{2}\right)$$

EXAMPLE 5 Using Substitution

Evaluate $\int x^2\sqrt{5+2x^3}\,dx$.

$u = x^2$
$du = 2x \ldots ?$

SOLUTION

This integral invites the substitution $u = 5 + 2x^3$, $du = 6x^2\,dx$.

$$\int x^2\sqrt{5+2x^3}\,dx. = \int (5+2x^3)^{1/2}\cdot x^2\,dx$$

$$= \frac{1}{6}\int (5+2x^3)^{1/2}\cdot 6x^2\,dx$$ Set up the substitution with a factor of 6.

$$= \frac{1}{6}\int u^{1/2}\,du$$ Substitute u for $5 + 2x^3$ and du for $6x^2\,dx$.

$$= \frac{1}{6}\left(\frac{2}{3}\right)u^{3/2} + C$$

$$= \frac{1}{9}(5+2x^3)^{3/2} + C$$ Re-substitute after antidifferentiating.

Now try Exercise 27.

EXAMPLE 6 Using Substitution

Evaluate $\int \cot 7x \, dx$.

$\checkmark$ look for this pattern!

$$\frac{u'}{u}$$

$$\int \cot 7x \, dx = \int \frac{\cos 7x}{\sin 7x} \, dx$$
Trigonometric identity

$$= \frac{1}{7} \int \frac{7 \cos 7x \, dx}{\sin 7x}$$
Note that $du = 7 \cos 7x \, dx$ when $u = \sin 7x$

We multiply by $\frac{1}{7} \cdot 7$, or 1.

$$= \frac{1}{7} \int \frac{du}{u}$$
Substitute u for $7 \sin x$ and du for $7 \cos 7x \, dx$.

$$= \frac{1}{7} \ln |u| + C$$
Notice the absolute value!

$$= \frac{1}{7} \ln |\sin 7x| + C$$
Re-substitute $\sin 7x$ for u after antidifferentiating.

Now try Exercise 29.

In Exercises 25–46, use substitution to evaluate the integral.

25. $\int \frac{dx}{(1-x)^2}$ $\quad \frac{1}{1-x} + C$

27. $\int \sqrt{\tan x}\, \sec^2 x\, dx$ $\quad \frac{2}{3}(\tan x)^{3/2} + C$

28. $\int \sec\left(\theta + \frac{\pi}{2}\right) \tan\left(\theta + \frac{\pi}{2}\right) d\theta$

29. $\int \tan(4x + 2)\, dx$

30. $\int 3(\sin x)^{-2}\, dx$

29. $-(1/4) \ln|\cos(4x+2)| + C$ or $(1/4) \ln|\sec(4x+2)| + C$

25

$u = 1 - x$

$du = -dx$

$-\int \frac{-dx}{(1-x)^2}$

$= -\int \frac{du}{u^2}$

$= -\int u^{-2} du$

$= -(-u^{-1})$

$= u^{-1} = \frac{1}{u}$

$= \frac{1}{1-x}$

27 $\quad u = \tan x$

$du = \sec^2 x\, dx$

$\int \sqrt{u}\, du$

$\int u^{1/2} du$

$\frac{u^{3/2}}{3/2} + C$

$\frac{2}{3} u^{3/2} + C$

$= \frac{2}{3}(\tan x)^{3/2} + C$

put back in terms of x!

29. $\displaystyle\int \tan(4x + 2)\, dx$

$u = \cos(4x+2)$

$du = -\sin(4x+2)\cdot 4$

$-\frac{1}{4}\,du = \sin(4x+2)$

$\displaystyle\int \frac{\sin(4x+2)}{\cos(4x+2)}$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$= \int -\frac{1}{4}\left(\frac{1}{u}\right)du$$

$$= -\frac{1}{4}\left(\ln|u|\right) + C$$

$$= -\frac{1}{4}\left(\ln\left|\cos(4x+2)\right|\right) + C$$

31. $\int \cos(3z+4)\,dz$ $\quad \frac{1}{3}\sin(3z+4)+C$

33. $\int \frac{\ln^6 x}{x}\,dx$ $\quad \frac{1}{7}(\ln x)^7 + C$

35. $\int s^{1/3}\cos(s^{4/3}-8)\,ds$ $\quad \frac{3}{4}\sin(s^{4/3}-8)+C$

37. $\int \frac{\sin(2t+1)}{\cos^2(2t+$ $\quad (1/2)\sec(2t+1)+C$

39. $\int \frac{dx}{x\ln x}$

41. $\int \frac{x\,dx}{x^2+1}$

31

$u = 3z+4$

$du = 3\,dz$

$\frac{1}{3}du = dz$

$\int \frac{1}{3}\cos(u)\,du$

$= \frac{1}{3}\sin(3z+4)+C$

37. $\displaystyle\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt$ [redacted] $\int$

(1/2) sec (2t + 1) + C

$\int \; dx$

$$\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta}$$

$$\int \tan(2t+1)\sec(2t+1)\,dt$$

$u = 2t+1$

$du = 2\,dt$

$\frac{1}{2}du = dt$

$$\frac{1}{2}\int \tan u \sec u \, du$$

$$= \frac{1}{2}\sec(2t+1) + C$$

33. $\int \frac{\ln^6 x}{x}\,dx$ $\quad \frac{1}{7}(\ln x)^7 + C$

$\frac{d}{dx}\ln x = \frac{1}{x}$

$\int \ln^6 x \cdot \frac{1}{x}\,dx$ $\quad u = \ln x$

$du = \frac{1}{x}\,dx$

$= \int u^6\,du$

$\frac{1}{7}u^7 + C = \frac{1}{7}(\ln x)^7 + C$

3

33. $\displaystyle\int \frac{\ln^6 x}{x}\,dx$ $\quad \frac{1}{7}(\ln x)^7 + C$

$u = \frac{1}{x}$ $\quad du = -x^{-2}\,dx$

$= x^{-1}$ $\quad$??

I need to substitute...

...this should be easier...

35. $\int s^{1/3} \cos(s^{4/3} - 8)\, ds$

$\frac{3}{4} \sin(s^{4/3} - 8) + C$

$u = s^{4/3} - 8$

$du = \frac{4}{3} s^{1/3}\, ds$

$\frac{3}{4} du = s^{1/3}\, ds$

$\int \frac{3}{4} \cos(u)\, du$

$= \frac{3}{4} \sin(s^{4/3} - 8) + C$

39. $\int \frac{dx}{x \ln x}$

41. $\int \frac{x\,dx}{x^2 + 1}$

(39)

$u = \ln x$

$du = \frac{1}{x}\,dx$

$\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot u' \quad \frac{u'}{u}$

$\int \frac{du}{u} = \int \frac{1}{u}\,du$

$= \ln u + C$

$= \ln|\ln x| + C$

(41)

$u = x^2 + 1$

$du = 2x\,dx$

$\frac{1}{2}\int \frac{2x\,dx}{x^2+1}$

$\frac{1}{2}\int \frac{du}{u}$

$\frac{1}{2} \ln u$

$\frac{1}{2} \ln|x^2+1|$

$= \frac{1}{2}$

39. $\int \frac{dx}{x \ln x}$

41. $\int \frac{x\,dx}{x^2 + 1}$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$= \int \frac{du}{u} = \int \frac{1}{u} \cdot du$

$= \ln u$

$= \ln(\ln(x)) + C$

41. $\displaystyle\int \frac{x\,dx}{x^2 + 1}$

EXAMPLE 7 **Setting Up a Substitution with a Trigonometric Identity**

Find the indefinite integrals. In each case you can use a trigonometric identity to set up a substitution.

(a) $\displaystyle\int \frac{dx}{\cos^2 2x}$ **(b)** $\displaystyle\int \cot^2 3x\, dx$ **(c)** $\displaystyle\int \cos^3 x\, dx$

$$\frac{d}{dx}[\tan u] = \sec^2 u \cdot u'$$

SOLUTION

(a) $\displaystyle\int \frac{dx}{\cos^2 2x} = \int \sec^2 2x\, dx$

EXAMPLE 7 Setting Up a Substitution with a Trigonometric Identity

Find the indefinite integrals. In each case you can use a trigonometric identity to set up a substitution.

(a) $\displaystyle\int \frac{dx}{\cos^2 2x}$ **(b)** $\displaystyle\int \cot^2 3x\, dx$ **(c)** $\displaystyle\int \cos^3 x\, dx$

(b) $\displaystyle\int \cot^2 3x\, dx = \int (\csc^2 3x - 1)\, dx$

EXAMPLE 7 **Setting Up a Substitution with a Trigonometric Identity**

Find the indefinite integrals. In each case you can use a trigonometric identity to set up a substitution.

(a) $\displaystyle\int \frac{dx}{\cos^2 2x}$ **(b)** $\displaystyle\int \cot^2 3x\, dx$ **(c)** $\displaystyle\int \cos^3 x\, dx$

In Exercises 47–52, use the given trigonometric identity to set up a u-substitution and then evaluate the indefinite integral.

47. $\int \sin^3 2x\,dx, \quad \sin^2 2x = 1 - \cos^2 2x$

48. $\int \sec^4 x\,dx, \quad \sec^2 x = 1 + \tan^2 x$

49. $\int 2\sin^2 x\,dx, \quad \cos 2x = 2\sin^2 x - 1$ $\quad x + \frac{\sin 2x}{2} + C$

$+1 \qquad +1$

$\cos 2x + 1 = 2\sin^2 x$

$\int (\cos 2x + 1)\,dx$

$u = 2x$

$du = 2\,dx$

50. $\int 4\cos^2 x\,dx, \quad \cos 2x = 1 - 2\cos^2 x$

$\frac{1}{2}\int 2(\cos 2x + 1)\,dx$

$= \frac{1}{2}\int (\cos u + 1)\,du$

$\frac{1}{2}(\sin u + u) = \frac{\sin 2x}{2} + \frac{2x}{2} + C$

EXAMPLE 8 Evaluating a Definite Integral by Substitution

Evaluate $\int_0^{\pi/3} \tan x \sec^2 x \, dx$.

In Exercises 53–66, make a u-substitution and integrate from $u(a)$ to $u(b)$.

53. $$\int_0^3 \sqrt{y+1}\, dy$$

54. $$\int_0^1 r\sqrt{1-r^2}\, dr$$

55. $$\int_{-\pi/4}^0 \tan x \sec^2 x\, dx$$

56. $$\int_{-1}^1 \frac{5r}{(4+r^2)^2}\, dr$$

C

57. $$\int_0^1 \frac{10\sqrt{\theta}}{(1+\theta^{3/2})^2}\, d\theta$$

58. $$\int_{-\pi}^{\pi} \frac{\cos x}{\sqrt{4+3\sin x}}\, dx$$

59. $$\int_0^1 \sqrt{t^5+2t}\,(5t^4+2)\, dt$$

60. $$\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\, d\theta$$

61. $$\int_0^7 \frac{dx}{x+2}$$

62. $$\int_2^5 \frac{dx}{2x-3}$$

63. $$\int_1^2 \frac{dt}{t-3}$$

64. $$\int_{\pi/4}^{3\pi/4} \cot x\, dx$$

65. $$\int_{-1}^3 \frac{x\, dx}{x^2+1}$$

66. $$\int_0^2 \frac{e^x\, dx}{3+e^x}$$

In Exercises 53–66, make a u-substitution and integrate from $u(a)$ to $u(b)$.

53. $\displaystyle\int_0^3 \sqrt{y+1}\, dy$ 14/3

55. $\displaystyle\int_{-\pi/4}^0 \tan x \sec^2 x\, dx$ $-1/2$

C

57. $\displaystyle\int_0^1 \frac{10\sqrt{\theta}}{(1+\theta^{3/2})^2}\, d\theta$ 10/3

59. $\displaystyle\int_0^1 \sqrt{t^5+2t}\,(5t^4+2)\, dt$ $2\sqrt{3}$

3/4

61. $\displaystyle\int_0^7 \frac{dx}{x+2}$ 1.504

63. $\displaystyle\int_1^2 \frac{dt}{t-3}$ -0.693

65. $\displaystyle\int_{-1}^3 \frac{x\, dx}{x^2+1}$ 0.805

$\frac{3}{2}x = 10$ $x = \frac{20}{3}$

57

$$\int_0^1 \frac{10\,\theta^{1/2}}{(1+\theta^{3/2})^2}\, d\theta$$

$u = 1+\theta^{3/2}$

$du = \frac{3}{2}\theta^{1/2}$

$\frac{20}{3} du = 10\theta^{1/2}$

$u = 1+0^{3/2}$
$= 1$

$u = 1+1^{3/2}$
$= 1+1 = 2$

$$\int_1^2 \frac{20}{3u^2}\, du$$ evaluate

63 $u = t-3$

$du = dt$

$u = 2-3 = -1$

$u = 1-3 = -2$

$$\int_{-2}^{-1} \frac{1\,du}{u}$$

$\ln|u| \Big|_{-2}^{-1}$ evaluate