6.3 Integration by Parts

What you'll learn about

- Product Rule in Integral Form
- Solving for the Unknown Integral
- Tabular Integration
- Inverse Trigonometric and Logarithmic Functions

Product Rule in Integral Form

When u and v are differentiable functions of x, the Product Rule for differentiation tells us that

$$\int \frac{d}{dx}(uv) = \int u\frac{dv}{dx} + \int v\frac{du}{dx}.$$

Integrating both sides with respect to x and rearranging leads to the integral equation

$$\int\left(u\frac{dv}{dx}\right)dx = \int\left(\frac{d}{dx}(uv)\right)dx - \int\left(v\frac{du}{dx}\right)dx$$

$$= uv - \int\left(v\frac{du}{dx}\right)dx.$$

When this equation is written in the simpler differential notation we obtain the following formula.

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du$$

EXAMPLE 1 Using Integration by Parts

Evaluate $\int x \cos x\,dx$.

SOLUTION

We use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = x, \qquad dv = \cos x\,dx.$$

To complete the formula, we take the differential of u and find the simplest antiderivative of $\cos x$.

$$du = dx \qquad v = \sin x$$

Then,

$$\int x \cos x\,dx = x \sin x - \int \sin x\,dx = x \sin x + \cos x + C.$$

Now try Exercise 1.

Q: is there an incorrect way to set this up?

A: Oh. Yes.

EXPLORATION 1 Choosing the Right u and dv

Not every choice of u and dv leads to success in antidifferentiation by parts. There is always a trade-off when we replace $\int u\,dv$ with $\int v\,du$, and we gain nothing if $\int v\,du$ is no easier to find than the integral we started with. Let us look at the other choices we might have made in Example 1 to find $\int x \cos x\,dx$.

1. Apply the parts formula to $\int x \cos x\,dx$, letting $u = 1$ and $dv = x \cos x\,dx$. Analyze the result to explain why the choice of $u = 1$ is never a good one.
2. Apply the parts formula to $\int x \cos x\,dx$, letting $u = x \cos x$ and $dv = dx$. Analyze the result to explain why this is not a good choice for this integral.
3. Apply the parts formula to $\int x \cos x\,dx$, letting $u = \cos x$ and $dv = x\,dx$. Analyze the result to explain why this is not a good choice for this integral.
4. What makes x a good choice for u and $\cos x\,dx$ a good choice for dv?

1 $u = 1$ $dv = x\cos x\,dx$

$du = 0$ $v =$ ☹

2 $u = x\cos x$ $dv = dx$

$du = (1)(\cos x)\,dx + (-\sin x)(x)$ $v = x$

$$\int u\,dv = uv - \int v\,du$$

$$= x^2 - \int x(\cos x)\,dx$$

In Exercises 1–10, find the indefinite integral.

$\int u\,dv = uv - \int v\,du$

1. $\int x \sin x \, dx$

$-x \cos x + \sin x + C$

2

3. $\int 3t\, e^{2t}\, dt$ $\quad \frac{3}{2} t e^{2t} - \frac{3}{4} e^{2t} + C$

4

$u = x \qquad dv = \sin x \, dx$

$du = dx \qquad v = -\cos x$

$= -x \cos x - \int (-\cos x)\, dx$

$= -x \cos x + \sin x \, dx$

$+ C$

3. $\int 3t\, e^{2t}\, dt$ $\quad \frac{3}{2} t e^{2t} - \frac{3}{4} e^{2t} + C$

$u = 3t \qquad dv = e^{2t}\, dt$

$du = 3\, dt \qquad v = \frac{1}{2} e^{2t}$

$= (3t)\left(\frac{1}{2} e^{2t}\right) - \int \frac{1}{2} e^{2t} (3)\, dt$

$\frac{3}{2} t e^{2t} - \frac{3}{4} e^{2t} + C$

EXAMPLE 2 Repeated Use of Integration by Parts

Evaluate $\int x^2 e^x \, dx$.

SOLUTION

With $u = x^2$, $dv = e^x \, dx$, $du = 2x \, dx$, and $v = e^x$, we have

$$\int x^2 e^x \, dx = x^2 e^x - 2\int x e^x \, dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x \, dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C.$$

Hence,

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2\int x e^x \, dx \\ &= x^2 e^x - 2x e^x + 2e^x + C. \end{aligned}$$

5. $\int x^2 \cos x \, dx$

$x^2 \sin x + 2x \cos x - 2 \sin x + C$

7. $\int 3x^2 e^{2x} \, dx$

9. $\int y \ln y \, dy$ $\quad \frac{y^2}{2} \ln y - \frac{y^2}{4} + C$

7. $\frac{3}{2} x^2 e^{2x} - \frac{3}{2} x e^{2x} + \frac{3}{4} e^{2x} + C$

(5)

$u = x^2 \quad dv = \cos x\,dx$

$du = 2x\,dx \quad v = \sin x$

$\int u\,dv = uv - \int v\,du$

$\int x^2 \cos x\,dx = \underline{x^2 \sin x} - \int 2x \sin x\,dx$

$u = 2x \quad dv = \sin x\,dx$

$du = 2 \quad v = -\cos x$

$\int 2x \sin x\,dx = \underline{2x(-\cos x)} - \int -2\cos x$

$\underline{+2\sin x}$

$= \underline{x^2 \sin x} \underline{+2x\cos x} \underline{-2\sin x} + C$

7. $\int 3x^2 e^{2x}\, dx$

$\frac{d}{dx} e^{2x} = 2 \cdot e^{2x}$

derive... anti

u		dv
$3x^2$		e^{2x}
	$+$	
$6x$		$\frac{1}{2} e^{2x}$
	$-$	
6		$\frac{1}{4} e^{2x}$
	$+$	
0		$\frac{1}{8} e^{2x}$

$\frac{3}{2} x^2 e^{2x}$

$-\frac{3}{2} x e^{2x}$

$\frac{3}{4} e^{2x}$

7. $\frac{3}{2}x^2e^{2x} - \frac{3}{2}xe^{2x} + \frac{3}{4}e^{2x} + C$

6. $\int x^2 e^{-x}\,dx$

$-x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$

$\int u\,dv = uv - \int v\,du$

$u = x^2 \quad dv = e^{-x}$

$du = 2x\,dx \quad v = -e^{-x}$

$= (x^2)(-e^{-x}) + \int 2x e^{-x}\,dx + C$

$u = 2x \quad dv = e^{-x}$

$du = 2\,dx \quad v = -e^{-x}$

$= -x^2e^{-x} + 2x(-e^{-x}) - \int (-e^{-x})(2\,dx)$

$= -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x}\,dx$

$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$

9. $\int y \ln y \, dy$ $\quad \dfrac{y^2}{2} \ln y - \dfrac{y^2}{4} + C$

u	dv
$\ln y$	y
$\frac{1}{x}$	$\frac{y^2}{2}$
$-\frac{1}{x^2}$	$\frac{y^3}{6}$

u	dv
y	$\ln y$

9. $\int y \ln y \, dy \quad \frac{y^2}{2}\ln y - \frac{y^2}{4} + C$

$u = \ln y \quad dv = y$

$du = \frac{1}{y} dy \quad v = \frac{y^2}{2}$

$uv - \int v \, du$

$(\ln y)\left(\frac{y^2}{2}\right) - \int \frac{y^{2}}{2} \frac{1}{y} dy$

$\frac{y^2 \ln y}{2} - \int \frac{y}{2} dy$

$= \frac{y^2 \ln y}{2} - \frac{1}{2}\left(\frac{y^2}{2}\right) + C$

$\equiv$

EXAMPLE 3 Solving an Initial Value Problem

Solve the differential equation $dy/dx = x \ln(x)$ subject to the initial condition $y = -1$ when $x = 1$. Confirm the solution graphically by showing that it conforms to the slope field.

SOLUTION

We find the antiderivative of $x \ln(x)$ by using parts. It is usually a better idea to differentiate $\ln(x)$ than to antidifferentiate it (do you see why?), so we let $u = \ln(x)$ and $dv = x\,dx$.

$$y = \int x \ln(x)\, dx$$

$$= \left(\frac{x^2}{2}\right)\ln(x) - \int \left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right) dx$$

$$= \left(\frac{x^2}{2}\right)\ln(x) - \int \left(\frac{x}{2}\right) dx$$

$$= \left(\frac{x^2}{2}\right)\ln(x) - \frac{x^2}{4} + C$$

Using the initial condition,

$$-1 = \left(\frac{1}{2}\right)\ln(1) - \frac{1}{4} + C$$

$y=-1,\ x=1\ \ldots$

$$-\frac{3}{4} = 0 + C$$

$$C = -\frac{3}{4}.$$

Thus

$$y = \left(\frac{x^2}{2}\right)\ln(x) - \frac{x^2}{4} - \frac{3}{4}.$$

Figure 6.9 shows a graph of this function superimposed on a slope field for $dy/dx = x \ln(x)$, to which it conforms nicely. ***Now try Exercise 11.***

In Exercises 11–16, solve the initial value problem. Confirm your answer by checking that it conforms to the slope field of the differential equation.

11. $\dfrac{dy}{dx} = (x + 2)\sin x$ and $y = 2$ when $x = 0$

$y = \int (x+2)\sin x$

$u = x+2 \quad dv = \sin x$

$du = dx \quad v = -\cos x$

$\cos$

$uv - \int v\,du$

$(x+2)(-\cos x) - \int (-\cos x)\,dx$

$-(x+2)(\cos x) + \sin x + C$

$y = -(x+2)(\cos 0) + \sin 0 + C$

$2 = -(2)(\cos 0) + \sin 0 + C$

$2 = -(2)(1) + 0 + C$

$C = 4$

$y = -(x+2)(\cos x) + \sin x + 4$

13. $\dfrac{du}{dx} = x\sec^2 x$ and $u = 1$ when $x = 0$

15. $\dfrac{dy}{dx} = x\sqrt{x - 1}$ and $y = 2$ when $x = 1$

EXAMPLE 4 Solving for the Unknown Integral

Evaluate $\int e^x \cos x\, dx$.

$$\int e^x \cos x\, dx$$

SOLUTION

Let $u = e^x$, $dv = \cos x\, dx$. Then $du = e^x\, dx$, $v = \sin x$, and

$$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx.$$

The second integral is like the first, except it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \quad dv = \sin x\, dx, \quad v = -\cos x, \quad du = e^x\, dx.$$

Then

$$\int e^x \cos x\, dx = e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x\, dx)\right)$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x\, dx.$$

$$\int e^x \cos x\, dx = e^x \sin x + e^x \cos x - \int e^x \cos x\, dx$$

$$+\int e^x \cos x\, dx \qquad\qquad +\int e^x \cos x\, dx$$

$$2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x$$

$$\boxed{\int e^x \cos x\, dx = \frac{e^x \sin x + e^x \cos x}{2}}$$

$$\frac{2}{2}$$

19. $\frac{e^x}{5}(2\sin 2x + \cos 2x) + C$

In Exercises 17–20, use parts and solve for the unknown integral.

17. $\int e^x \sin x\, dx$

$\frac{e^x}{2}(\sin x - \cos x) + C$

19. $\int e^x \cos 2x\, dx$ $=$

$u = e^x \quad dv = \cos 2x$

$du = e^x dx \quad v = \sin 2x \cdot \frac{1}{2}$

$\frac{1}{2} e^x \sin 2x - \int \frac{1}{2} \sin 2x \cdot e^x dx$

$u = e^x \quad dv = \sin 2x$

$du = e^x dx \quad v = -\cos 2x \cdot \frac{1}{2}$

$e^x(-\cos 2x)\frac{1}{2} - \int (-\cos 2x)\frac{1}{2} e^x dx$

$-\frac{1}{2} e^x \cos 2x + \frac{1}{2} \int \cos 2x\, e^x dx$

In Exercises 17–20, use parts and solve for the unknown integral.

17. $\int e^x \sin x \, dx$ $\quad \frac{e^x}{2}(\sin x - \cos x) + C$

18. $\int e^{-x} \cos x \, dx$ $\quad \frac{e^{-x}}{2}(\sin x - \cos x) + C$

19. $\int e^x \cos 2x \, dx$

20. $\int e^{-x} \sin 2x \, dx$

17

$u = e^x \quad dv = \sin x \, dx$

$du = e^x dx \quad v = -\cos x$

$= (e^x)(-\cos x) - \int (-\cos x)(e^x) dx$

$= -e^x \cos x + \int \cos x \, e^x dx$

$u = e^x \quad dv = \cos x \, dx$

$du = e^x \quad v = \sin x$

$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int \sin x \, e^x dx$

$+\int e^x \sin x \, dx$ "chasing the tail!" $\Rightarrow$ $+\int e^x \sin x \, dx$

$\frac{2\int e^x \sin x \, dx}{2} = \frac{-e^x \cos x}{2} + \frac{e^x \sin x}{2} + \frac{C}{2}$

$\int e^x \sin dx = -\frac{1}{2} \cos x + \frac{1}{2} e^x \sin x + C$

EXAMPLE 5 Using Tabular Integration

Evaluate $\int x^2 e^x\,dx$.

SOLUTION

With [redacted]) $= x^2$ and [redacted]) $= e^x$, we list:

u dv

dv

[redacted]) and its derivatives		[redacted]) and its integrals
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

u

$\sin x$
$\cos x$
$-\sin x$
$-\cos x$

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x\,dx = x^2 e^x - 2xe^x + 2e^x + C.$$

Compare this with the result in Example 2. *Now try Exercise 21.*

EXAMPLE 6 Using Tabular Integration

Evaluate $\int x^3 \sin x \, dx$.

SOLUTION

With $f(x) = x^3$ and $g(x) = \sin x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

Again we combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.$$

Now try Exercise 23.

In Exercises 21–24, use tabular integration to find the antiderivative.

21. $\int x^4 e^{-x}\,dx$

$(-x^4 - 4x^3 - 12x^2 - 24x - 24)e^{-x} + C$

23. $\int x^3 e^{-2x}\,dx$

23. $\left(-\frac{x^3}{2} - \frac{3x^2}{4} - \frac{3x}{4} - \frac{3}{8}\right)e^{-2x} + C$

$uv \mp \int v\,du$

23

	u	dv
+	x^3	e^{-2x}
−	$3x^2$	$-\frac{1}{2}e^{-2x}$
+	$6x$	$\frac{1}{4}e^{-2x}$
−	6	$-\frac{1}{8}e^{-2x}$
	0	$\frac{1}{16}e^{-2x}$

$x^3\left(-\frac{1}{2}e^{-2x}\right) - (3x^2)\left(\frac{1}{4}e^{-2x}\right)$

21. $\int x^4 e^{-x}\,dx$

$(-x^4 - 4x^3 - 12x^2 - 24x - 24)e^{-x} + C$

u		du
x^4		e^{-x}
	$+$	$-e^{-x}$
$4x^3$	$-$	e^{-x}
$12x^2$	$+$	$-e^{-x}$
$24x$	$-$	e^{-x}
24	$+$	$-e^{-x}$
0		

$(x^4)(-e^{-x}) - (4x^3)(e^{-x})$

$+ (12x^2)(-e^{-x})$

$- (24x)(e^{-x})$

$+ (24)(-e^{-x})$